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Estimation Techniques for Censored Sample Data in Accelerated Life Testing

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Abstract

Common in medical studies or reliability analysis, the failure of individuals or units may be attributable to more than one cause. Also, design the life-testing experiments under a highly reliable product or materials required a long period of time which requires stress higher than normal stress. In this paper, I adopt the mixture of two exponential distributions which has become increasingly necessary in engineering statistics. Under a step stress accelerate life tests and type II censoring method for independent competing risks data the model is formulated. The estimation problem is addressed here from a classical viewpoint. The maximum likelihood estimates of the unknown model parameters are formulated. Additionally, the asymptotic confidence intervals depend on normality theory, along with the two bootstrap confidence intervals are proposed. Finally, to validate the proposed model and assess the estimation methods, I formulated an extensive simulation study.

Keywords: Mixture of two distributions, Accelerate life tests; Competing risks model, MLEs, Bootstrap confidence intervals.

1. INTRODUCTION

Increasing technological advancements lead to the demand for highly reliable and long-lasting solutions. Technology is expected to work without failure and adapt to the changing world, which challenges technical abilities to reliable analysis, see [1]. Mixed models are created to support diversity in applications, technological advancements, implementing new methodologies and providing a wider perspective to computational tools. These models have been adopted into multiple branches of statistics, notably using a mixture of distribution for hazard models, see [2]. Mixture distributions consist of multiple components (finite or infinite) describing different data features. They provide a detailed insight into the complex systems thus adopted into various fields like engineering, information technology, bioinformatics, biostatistics, ecology, and robotics, see [3]. Determining random and fixed effects of distribution is a critical challenge when modifying such a model according to data. This issue can be investigated by assessing the nullification of a sample of random effects through its deviations. Random effect has characteristics varying on individual

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basis, while fixed has same characteristics for individuals, see [4]. When models are either known or limited to an unknown distribution confined by given constraints, the inference is calculated using likelihood maximization, see [5]. The findings of the likelihood-based evaluation can also be used to determine starting values for chains and to verify that the outcomes are reasonable, see [6].

Finding a distribution that fits a given collection of data is one of the fascinating challenges in statistics. The goal is to see if a particular distribution matches the data. In real life experiments, usually some data points withdrew from the experience which causes the occurrence of censored data. The censored data is utilized when some but not all datapoints under the test failed through a specific period, see [7]. The censoring of type I and type II are the most common types of censoring data. In type I censoring, the number of failures observed is a random variable, whereas the experiment time is static. As opposite, in type II censoring, the experiment time is random and the observed failure times are static. Apart from these two types, a combination of both can also be used, namely, hybrid censoring technique or mixed censoring. The most used censoring technique is type II censoring and has a time-to-death model to estimate dependability, see [8]. Data sets with censored data can include single or multiply censored data depending on the data. The literature on the reliability of mixed models with type II censoring is still scarce, see [9]. Because the difficulty of simulating field failure processes in a laboratory environment is critical for expedited lifespan estimate, products are subjected to harsher-than-normal conditions to capture failure data successfully, see [10]. Solutions like accelerated life testing are introduced to speed up the failure process without changing the existing failure mechanism. This is one of the high-stress testing techniques that is done by exposing the product to extreme conditions, see [11]. Multiple ALTs types have been introduced over the years, i.e., constant-stress, step-stress and progressive-stress. As given by [12], and [13], the maximum work carried out in single stress ALTs, which focuses on model parameters with statistical inference. Most accelerated life-test investigations nowadays concentrate on a specific component or failure mechanism [14]. As a result, products with various features or failure mechanisms face difficulties in ensuring product dependability at the system level [15]. The population's target sample should be established from the start, as it influences the distribution decision. Several methods for modeling and analysing ALT data for reliability demonstration are offered and statistics-centered methods are the most significantly utilized models [16]. Multiple stress ALT models and type II data filtering models have been presented over the years to improve system dependability and reduce failures. The lifetime of the product is according to Weibull distribution, and it can be applied using other distributions to increase the project results [17]. The sample data for ALT is difficult to find due to the time that it takes for the product life cycle and the extent to which every aspect of the product is fully used. Data-driven models rely on probabilistic distributions like most statistical models. Thus, whether to use Monte Carlo or adaptive filtering technique require an ample amount of data sets to implement these models for accuracy. As per the Lithium-Ion case study, the data set covers a wide range of characteristics of the battery life, and models are developed on those data sets to create a reliable evaluation of its life cycle, see [18]. It further mentions that the wider and unique the data set, the higher the precision of the prediction model is achievable, see [19]. Hence, to study ALT methods, obtaining data samples of the performance is necessary when using the model-based evaluation [20]. The unknown parameters of an external factor still rely on a huge number of ALT results; therefore, multiple tests are carried out to find the root cause of failure, see [21]. The findings indicated that observed lifetime tests at the lab level are crucial to ensure that the likely execution and lifetime conditions can be gathered in actual field procedures. To anticipate the lifetime, collect exact and complete test data, analyse, and process it, see [22]. ALT method is used to achieve the reliability test objective under the assumption of restricted test time and money to assess the lifetime product fast and precisely, see [23]. The product's lifetime is supposed to follow a given distribution type for ALT data modeling. A matching statistical model of the product's physical failure process is utilized to assess and evaluate the reliability of the information obtained under accelerated stress, see [24], and [25].

To assess the product's lifetime in a quick and precise way, ALT scheme is used to arrive at the target of the reliability test under the basis of the limited test time, see [26]. For the purpose of studying ALT data modeling, the product's lifetime is taken to follow a particular distribution type. A matching statistics model of the product's physical failure mechanism is accustomed to analyze and convert the reliability information under accelerated stress obtaining its reliability information under normal stress, see [26], and [27]. Using the Expectation maximization model, Peng and Xu [28], achieved the distribution of parameters, but due to its complexity later, [29] proposed the use of Markov Chain Monte Carlo to achieve the distribution type.

In practice, when the failure of individuals or materials occur under different causes of failure then, I mean competing risk models. In this model, I measure the risk of one reason with respect to other reasons of failure. This problem discussed early by [30], and recently by [31], and [32], for partially observed causes of failure. Also, this problem discussed under joint samples by [31], [33], and [34]. I aim in this article to proposes a mixture modeling of exponential distributions to analyze the competing risks data under multiple independent causes of failure. For saving time and cost, I applied step-stress partially accelerated life tests with type II censoring scheme. Additionally, I combine the type II censoring with competing risk model to estimate the risk of a cause of failure with respect to other independent causes of failures. Therefore, the observed data are processed to estimate the model parameters by applying different schemes of estimation. The maximum likelihood is used to obtain the point estimators. However, the asymptotic confidence intervals under normality theorem and two bootstrap confidence intervals are formulated. Estimators tested under Monte Carlo simulation study. This paper can serve, Engineer, reliability Engineers, Practitioners, Researchers and Academics in Reliability Engineering, System Safety Professionals, Industrial Engineers, Risk Analysts.

The structured of this article is summarized as following: In Section 2, the competing risks model is formulated under under type II censoring scheme. In section 3, the mixed model of two exponential distributions is formulated. In section 4: model description and its assumptions are presented. In Section 5, I formulated the maximum likelihood estimate of the model parameters. The asymptotic confidence intervals under normality theorem is formulated in Section 6. The two bootstrap confidence intervals are presented in Section 7. Estimation results are assessed and compared with Monte Carlo studying in Section 8. Finlay, Some comments are reported in conclusion in Section 9.

2. COMPETING RISKS MODEL WITH TYPE II CENSORING DATA

Suppose, a random sample of size n of items are placed into a life testing experiment. Earlier the experiment is running, number of failure m is determined. In type II censoring scheme, the joint likelihood function of m failure times $X_1, < \cdots < X_m$, is given by (Chapter 7, [35]):

$$f_{1,2,\ldots,m}(\mathbf{x}) = \frac{n!}{(n-m)}[1 - F(x_m)]^{n-m} \prod_{i=1}^{m} f(x_i), \tag{1}$$

Suppose that, the failure is done with respected to one of two independent causes of failure which is known by competing risks model. Therefore, the type II censoring scheme under competing risks model described as follows:

when the 1^{st} failure X_1 is observed the corresponding cause of failure, δ_1 is determined. Also, at the 2^{nd} failure X_2 is observed the corresponding cause of failure, δ_2 is determined. The experiment is continual until m-th failure X_m and its cause of failure δ_m are observed. The data (X_1, δ_1), (X_2, δ_2), ..., (X_m, δ_m) where $\delta_i = \{1, 2\}$, $i = 1, 2, ..., m$ is called type II competing risks sample. The value $\omega_i = 1$ means that, failure done with respected to the 1^{st} cause of failure and $\delta_i = 2$ means the failure under 2^{nd} cause. The joint likelihood function under type II competing risks data is given by:

$$f_{1,2,\ldots,m}(\mathbf{x}) = \frac{n!(S_1(x_m)S_2(x_m))^{(n-m)}}{(n-m)!} \prod_{i=1}^{m} S_1(x_i)S_2(x_i)(h_1(x_i))^{z(\delta_i=1)}(h_2(x_i))^{z(\delta_i=2)}, \tag{2}$$

where

$$z(\delta_i = j) = \begin{cases} 1, \ \delta_i = j \\ 0, \ \delta_i \neq j, \end{cases} \text{ for } \ j = 1, 2, \tag{3}$$

where, $S_j(.)$ is the survival function “SF”, and $h_j(.)$ is hazard failure rate “HRF”. FIGURE 1 shows a diagrammatic representation of the data production which leads to an example of the real type II censored data from an experiment. In the diagram the productions are affected by different failure causes.

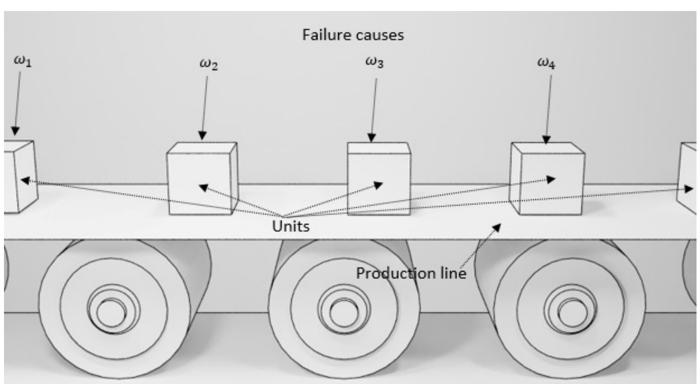


Figure 1: Diagrammatic and real type-II censored data from an experiment. The diagram shows different units going through production line.

3. MIXED LIFETIME MODEL

In this section, I consider the mixture distribution which is weighted as a summation of K one parameter distributions $g_1(x;\lambda_1),\ldots,g_K(x;\lambda_K)$. This distribution is formulated by

$$f(x;\lambda_1,\ldots,\lambda_K)=\sum_{i=1}^{K}\zeta_i g(x;\lambda_i), \tag{4}$$

where the weights satisfy $\sum_{i=1}^{K}\zeta_i=1$, and each distributions from this mixture has one parameter λ_i. In this article, I consider the mixture of two one-dimensional exponential distributions as an example for mixture of two continuous distributions given by:

$$g_1(x;\lambda_1)\sim exp(\lambda_1), \tag{5}$$
$$g_2(x;\lambda_2)\sim exp(\lambda_2), \tag{6}$$

the PDF is given as follows:

$$g(x;\lambda)=\lambda\exp\{-\lambda x\};x,\lambda>0, \tag{7}$$

and

$$f(x;\lambda_1,\lambda_2,\zeta)=\zeta g_1(x;\lambda_1)+(1-\zeta)g_2(x;\lambda_2)=\zeta\lambda_1\exp\{-\lambda_1 x\}+(1-\zeta)\lambda_2\exp\{-\lambda_2 x\}. \tag{8}$$

The cumulative distribution function CDF will take the following form:

$$F(x;\lambda_1,\lambda_2,\zeta)=1-\{\zeta exp\{-\lambda_1 x\}+(1-\zeta)\exp\{-\lambda_2 x\}\}, \tag{9}$$

and the survival function:

$$S(x;\lambda_1,\lambda_2,\zeta)=\zeta exp\{-\lambda_1 x\}+(1-\zeta)\exp\{-\lambda_2 x\}, \tag{10}$$

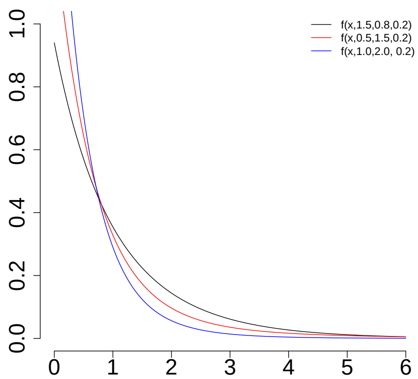
then the hazard function will take the following form:

$$h(x;\lambda_1,\lambda_2,\zeta)=\frac{\zeta\lambda_1\exp\{-\lambda_1 x\}+(1-\zeta)\lambda_2\exp\{-\lambda_2 x\}}{\zeta exp\{-\lambda_1 x\}+(1-\zeta)\exp\{-\lambda_2 x\}}, \tag{11}$$

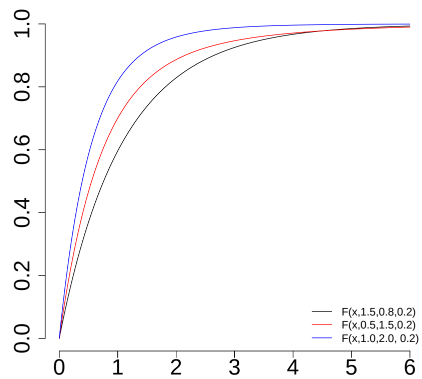
FIGURE 2 indicates the plots of the PDF, the CDF, the survival, and the hazard rate functions for the value of $\zeta=0.2$ and different scale parameters for a mixture of two one-dimensional exponential distributions.

4. MODEL DESCRIPTION AND ASSUMPTION

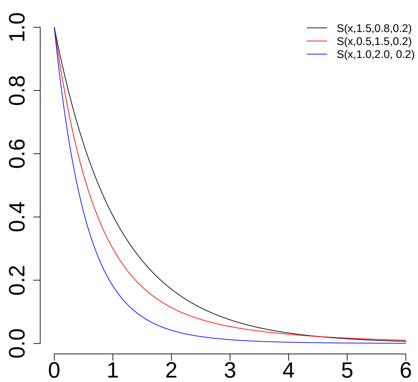
Assume, a sample of size n from identical independent items are randomly chosen from a mixture distribution with PDF and CDF given by Equations (8) and (9). These items are subject to a life testing experiment. Prior the experiment is running, two values m and t are determined, where m denotes the effect sample size needing for statistical inference, and t represents the stress change time. The failure time X_{ji} is defined as the life time of the i^{th} item under cause $j=1,2$. The observed failure times is defined by $X_i=\min\{X_{1i},X_{2i}\}$, i =1, 2, ..., m. Under step-stress partially



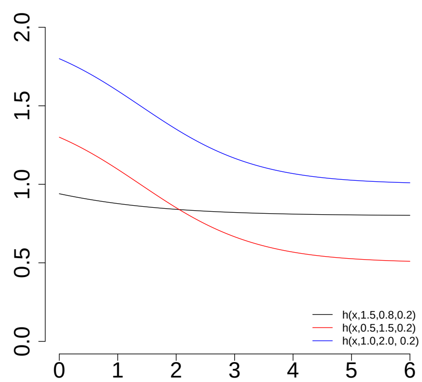
(a) Probability density functions.



(b) Cumulative distribution functions.



(c) Survival functions.



(d) Hazard functions.

Figure 2: Plots of the PDF, the CDF, the survival, and the hazard rate functions for the value of $\zeta = 0.2$ and different scale parameters.

ALT, all the sample of size n are put under normal stress conditions. At the time of the experiment is running, the failure time and the corresponding cause of failure are recorded until the stress changes time t is reached. Then, the n samples are placed under stress condition until $m-th$ failure is observed. The reduction in the test unit's lifetime appears to be proportional to the inverse of the acceleration factor, with the proportionality constant equal to the unit's remaining lifetime. The total lifetime of a test unit, denoted by $W_t(X_i)$, using accelerated conditions. The life time of a unit in partially step-stress ALTs, is define as follows:

$$W = \begin{cases} X_i, & X < t \\ t + \gamma^{-1}(X - t), & X > t, \end{cases} \tag{12}$$

where γ is an acceleration factor, conducts a decrease in the whole lifespan of the tested components and t is a changing time. Then, for a combination of two continuous distribution parameters λ_1, λ_2, and acceleration factor γ, the PDF, takes the following form,

$$g(w) = \begin{cases} 0, & \text{if } w < 0 \\ f_1(w), & \text{if } 0 < w < t, \\ f_2(w), & \text{if } w > t, \end{cases} \tag{13}$$

where $f_1(w)$ is given by Equation (8) and $f_2(w)$ is obtained from Equation (8) after transforming variable. The PDF, CDF, survival and hazard rate functions under stress conditions, respectively given by:

$$f_2(w; \lambda_1, \lambda_2, \zeta) = \zeta\gamma\lambda_1 \exp\{-\lambda_1(\gamma(w-t)+t)\} + (1-\zeta)\gamma\lambda_2 \exp\{-\lambda_2(\gamma(w-t)+t)\}, \tag{14}$$

$$F_2(w; \lambda_1, \lambda_2, \zeta, \gamma) = 1 - \{\zeta \exp\{-\lambda_1(\gamma(w-t)+t)\} + (1-\zeta) \exp\{-\lambda_2(\gamma(w-t)+t)\}\}, \tag{15}$$

$$S_2(w; \lambda_1, \lambda_2, \zeta, \gamma) = \{\zeta \exp\{-\lambda_1(\gamma(w-t)+t)\} + (1-\zeta) \exp\{-\lambda_2(\gamma(w-t)+t)\}\}, \tag{16}$$

the hazard function:

$$h_2(w_i; \lambda_{j2}, \sigma, \zeta_1, \gamma) = \frac{\zeta\gamma\lambda_1 \exp\{-\lambda_1(\gamma(w-t)+t)\} + (1-\zeta)\gamma\lambda_2 \exp\{-\lambda_2(\gamma(w-t)+t)\}}{\{\zeta \exp\{-\lambda_1(\gamma(w-t)+t)\} + (1-\zeta) \exp\{-\lambda_2(\gamma(w-t)+t)\}\}}. \tag{17}$$

Suppose that, the number of items which fail under normal conditions is represented by r, and number of failed items under accelerated conditions is denoted by $m-r$. Then, the type II censored random sample of the total lifetime W, defined, separately, in two case:

1- If $W_m < t$ then experiment is running completely under normal conditions and the joint likelihood function of censored sample $(W_1, \delta_1) < (W_2, \delta_2) < \cdots < (W_m, \delta_m) < t$ is given by:

$$f(w) = \frac{n!(S_{11}(w_m)S_{21}(w_m))^{(n-m)}}{(n-m)!} \prod_{i=1}^{m} S_{11}(w_i)S_{21}(w_i)(h_{21}(w_i))^{z(\delta_i=1)}(h_{21}(w_i))^{z(\delta_i=2)}, \tag{18}$$

and accelerated factor is equal to zero.

2- If $W_m > t$ the whole lifetime W is running under stress conditions, defined by: $(W_1, \delta_1) < (W_2, \delta_2) < \cdots < (W_r, \delta_r) < t < (W_{r+1}, \delta_{r+1}) < \cdots < (W_m, \delta_m)$. In this case the joint likelihood function

$$f(w) = \frac{n!(S_{12}(w_m)S_{22}(w_m))^{(n-m)}}{(n-m)!} \prod_{i=1}^{r} S_{11}(w_i)S_{21}(w_i)(h_{11}(w_i))^{z(\delta_i=1)}(h_{21}(w_i))^{z(\delta_i=2)}$$
$$\times \prod_{i=r+1}^{m} S_{12}(w_i)S_{22}(w_i)(h_{12}(w_i))^{z(\delta_i=1)}(h_{22}(w_i))^{z(\delta_i=2)} \tag{19}$$

$$z(\delta_i = j) = \begin{cases} 1, & \text{if } \delta_i = j \\ 0, & \text{if } \delta_i \neq j \end{cases}$$

3- The time-to-failure W_i, i =1, 2, , ..., m is taken to be $W_i = \min\{W_{1i}, W_{2i}\}$ and $j = 1, 2$.

4- The failure time W_{ji} has a mixed exponential lifetime distribution with CDF, defined by:

$$F_{j1}(w) = 1 - \left\{\zeta \exp\left\{-\lambda_{j1} w\right\} + (1-\zeta)\exp\left\{-\lambda_{j2} w\right\}\right\}. \tag{20}$$

and

$$F_{j2}(w) = 1 - \left\{\zeta \exp\left\{-\lambda_{j1}(\gamma(w-t)+t)\right\} + (1-\zeta)\exp\left\{-\lambda_{j2}(\gamma(w-t)+t)\right\}\right\}, \tag{21}$$

where $j = 1, 2$, denotes the causes of failure.

5. MAXIMUM LIKELIHOOD ESTIMATION

The corresponding joint likelihood function of mixture of two exponential lifetime distributions under the assumption of the observed type II censored sample, are defined as follows: $(W_1, \delta_1) < (W_2, \delta_2) < \cdots < (W_r, \delta_r) < t < (W_{r+1}, \delta_{r+1}) < \cdots < (W_m, \delta_m)$, given by Equation (19), is reduced to:

$$\begin{aligned}
L(\Theta|\mathbf{w}) = &\left\{\zeta^2 \exp\{-(\lambda_{11}+\lambda_{21})(\gamma(w_m-t)+t)\} + (1-\zeta)^2 \exp\{-(\lambda_{12}+\lambda_{22})(\gamma(w_m-t)+t)\}\right.\\
&+ \zeta(1-\zeta)\exp\{-(\lambda_{11}+\lambda_{21})(\gamma(w_m-t)+t)\}\\
&\left.+ \zeta(1-\zeta)\exp\{-(\lambda_{12}+\lambda_{22})(\gamma(w_m-t)+t)\}\right\}^{(n-m)}\\
&\times \prod_{i=1}^{r} \left\{\zeta^2 \lambda_{11}\exp\{-(\lambda_{11}+\lambda_{21})w_i\} + (1-\zeta)^2\lambda_{12}\exp\{-(\lambda_{12}+\lambda_{22})w_i\}\right.\\
&\left.+ \zeta(1-\zeta)\lambda_{11}\exp\{-(\lambda_{11}+\lambda_{22})w_i\} + \zeta(1-\zeta)\lambda_{12}\exp\{-(\lambda_{12}+\lambda_{21})w_i\}\right\}^{z(\delta_i=1)}\\
&\times \left\{\zeta^2\lambda_{21}\exp\{-(\lambda_{11}+\lambda_{21})w_i\} + (1-\zeta)^2\lambda_{22}\exp\{-(\lambda_{12}+\lambda_{22})w_i\}\right.\\
&\left.+ \zeta(1-\zeta)\lambda_{21}\exp\{-(\lambda_{21}+\lambda_{12})w_i\} + \zeta(1-\zeta)\lambda_{22}\exp\{-(\lambda_{11}+\lambda_{22})w_i\}\right\}^{z(\delta_i=2)}\\
&\times \prod_{i=r+1}^{m} \left\{\zeta^2\lambda_{11}\exp\{-(\lambda_{11}+\lambda_{21})(\gamma(w_i-t)+t))\}\right.\\
&+ (1-\zeta)^2\lambda_{12}\exp\{-(\lambda_{12}+\lambda_{22})(\gamma(w_i-t)+t))\}\\
&+ \zeta(1-\zeta)\lambda_{11}\exp\{-(\lambda_{11}+\lambda_{22})(\gamma(w_i-t)+t))\}\\
&\left.+ \zeta(1-\zeta)\lambda_{12}\exp\{-(\lambda_{12}+\lambda_{21})(\gamma(w_i-t)+t))\}\right\}^{z(\delta_i=1)}\\
&\times \left\{\zeta^2\lambda_{21}\exp\{-(\lambda_{11}+\lambda_{21})(\gamma(w_i-t)+t))\}\right.\\
&+ (1-\zeta)^2\lambda_{22}\exp\{-(\lambda_{12}+\lambda_{22})(\gamma(w_i-t)+t))\}\\
&+ \zeta(1-\zeta)\lambda_{21}\exp\{-(\lambda_{21}+\lambda_{12})(\gamma(w_i-t)+t))\}\\
&\left.+\zeta(1-\zeta)\lambda_{22}\exp\{-(\lambda_{11}+\lambda_{22})(\gamma(w_i-t)+t))\}\right\}^{z(\delta_i=2)}
\end{aligned} \tag{22}$$

The comprehending log-likelihood function is given by:

$$\begin{aligned}
l(\Theta|\mathbf{w}) = &(n-m)\log\left\{\zeta^2\exp\{-(\lambda_{11}+\lambda_{21})(\gamma(w_m-t)+t)\}\right.\\
&+(1-\zeta)^2\exp\{-(\lambda_{12}+\lambda_{22})(\gamma(w_m-t)+t)\}\\
&+ \zeta(1-\zeta)\exp\{-(\lambda_{11}+\lambda_{21})(\gamma(w_m-t)+t)\}
\end{aligned}$$

$$
\begin{aligned}
&+\zeta(1-\zeta)\exp\{-(\lambda_{12}+\lambda_{22})(\gamma(w_m-t)+t)\}\}\\
&+\sum_{i=1}^{r}\left\{\zeta^2\lambda_{11}\exp\{-(\lambda_{11}+\lambda_{21})w_i\}+(1-\zeta)^2\lambda_{12}\exp\{-(\lambda_{12}+\lambda_{22})w_i\}\right.\\
&\left.+\zeta(1-\zeta)\lambda_{11}\exp\{-(\lambda_{11}+\lambda_{22})w_i\}+\zeta(1-\zeta)\lambda_{12}\exp\{-(\lambda_{12}+\lambda_{21})w_i\}\right\}\\
&+\sum_{i=1}^{r}z(\delta_i=1)\left\{\zeta^2\lambda_{21}\exp\{-(\lambda_{11}+\lambda_{21})w_i\}+(1-\zeta)^2\lambda_{22}\exp\{-(\lambda_{12}+\lambda_{22})w_i\}\right.\\
&\left.+\zeta(1-\zeta)\lambda_{21}\exp\{-(\lambda_{21}+\lambda_{12})w_i\}+\zeta(1-\zeta)\lambda_{22}\exp\{-(\lambda_{11}+\lambda_{22})w_i\}\right\}\\
&+\sum_{i=r+1}^{m}z(\delta_i=1)\left\{\zeta^2\lambda_{11}\exp\{-(\lambda_{11}+\lambda_{21})(\gamma(w_i-t)+t))\}\right.\\
&+(1-\zeta)^2\lambda_{12}\exp\{-(\lambda_{12}+\lambda_{22})(\gamma(w_i-t)+t))\}\\
&+\zeta(1-\zeta)\lambda_{11}\exp\{-(\lambda_{11}+\lambda_{22})(\gamma(w_i-t)+t))\}\\
&\left.+\zeta(1-\zeta)\lambda_{12}\exp\{-(\lambda_{12}+\lambda_{21})(\gamma(w_i-t)+t))\}\right\}\\
&+\sum_{i=r+1}^{m}z(\delta_i=2)\left\{\zeta^2\lambda_{21}\exp\{-(\lambda_{11}+\lambda_{21})(\gamma(w_i-t)+t))\}\right.\\
&+(1-\zeta)^2\lambda_{22}\exp\{-(\lambda_{12}+\lambda_{22})(\gamma(w_i-t)+t))\}\\
&+\zeta(1-\zeta)\lambda_{21}\exp\{-(\lambda_{21}+\lambda_{12})(\gamma(w_i-t)+t))\}\\
&\left.+\zeta(1-\zeta)\lambda_{22}\exp\{-(\lambda_{11}+\lambda_{22})(\gamma(w_i-t)+t))\}\right\}
\end{aligned}
\tag{23}
$$

MLEs

The model parameters' point estimate are got by differentiate either the likelihood function as in Equation (22) or the log-likelihood equation as in Equation (23). Here, differentiating the likelihood function is complicated and I will focus on finding the derivative by using the log-likelihood function. The estimation of parameters $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}$, and γ can be achieved by differentiating the log-likelihood function with respect to these parameters. The derivatives these parameters are shown as follows:

$$\frac{\partial l(\Theta|\mathbf{w})}{\partial \lambda_{j1}} = 0,\ j = 1, 2, \tag{24}$$

$$\frac{\partial l(\Theta|\mathbf{w})}{\partial \lambda_{j2}} = 0,\ j = 1, 2, \tag{25}$$

and

$$\frac{\partial l(\Theta|\mathbf{w})}{\partial \gamma} = 0,\ j = 1, 2. \tag{26}$$

Different iteration method can be used to solve Equations (24), (25) and (26) to obtain the estimation of parameters $\hat{\Theta} = \{\hat{\lambda}_{11}, \hat{\lambda}_{12}, \hat{\lambda}_{21}, \hat{\lambda}_{22}, \hat{\gamma}\}$ which maximize the likelihood function.

6. APPROXIMATE CONFIDENCE INTERVALS

The matrix of Fisher information occurs from the minus expectation of the second partial derivative of log of the likelihood function that is denoted by κ. Interval estimation of parameters $\Theta = \{\lambda_{11}$, λ_{12}, λ_{21}, λ_{22}, $\gamma\}$ are found from asymptotic normality distribution of $\hat{\Theta} = \{\hat{\lambda}_{11}$, $\hat{\lambda}_{12}$, $\hat{\lambda}_{21}$, $\hat{\lambda}_{22}$, $\hat{\gamma}\}$ with mean $\Theta = \{\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \gamma\}$ and variance-covariance matrix κ^{-1}. This approximation is shown by the estimate of the following:

$$\hat{\Theta} = \{\hat{\lambda}_{11}, \hat{\lambda}_{12}, \hat{\lambda}_{21}, \hat{\lambda}_{22}, \hat{\gamma}\} \longrightarrow N(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \gamma\}, \kappa^{-1}) \tag{27}$$

The variance-covariance matrix κ_0^{-1} can be defend as

$$\kappa_0^{-1} = \begin{pmatrix} -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{11}^2} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{11}\lambda_{12}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{11}\lambda_{21}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{11}\lambda_{22}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{11}\gamma} \\ -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{12}\lambda_{11}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{12}^2\lambda_{12}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{12}\lambda_{21}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{12}\lambda_{22}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{12}\gamma} \\ -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{21}\lambda_{11}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{21}\lambda_{12}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{21}^2} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{21}\lambda_{22}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{21}\gamma} \\ -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{22}\lambda_{11}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{22}\lambda_{12}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{22}\lambda_{21}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{11}^2} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{22}\gamma} \\ -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{22}\lambda_{11}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{22}\lambda_{12}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{22}\lambda_{21}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{11}^2} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\lambda_{22}\gamma} \\ -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\gamma\lambda_{11}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\gamma\lambda_{12}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\gamma\lambda_{21}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\gamma\lambda_{22}} & -\frac{\partial^2 l(\Theta|\mathbf{w})}{\partial\gamma^2} \end{pmatrix}_{\hat{\Theta}=\{\hat{\lambda}_{11},\hat{\lambda}_{12},\hat{\lambda}_{21},\hat{\lambda}_{22},\hat{\gamma}\}}.$$

Therefore, $(1-\alpha)100\%$ confidence intervals of the model parameters is constructed by:

$$\begin{cases} \hat{\gamma} \mp z_{\frac{\alpha}{2}} e_{55}, \\ \hat{\lambda}_{11} \mp z_{\frac{\alpha}{2}} e_{11},\ \hat{\lambda}_{12} \mp z_{\frac{\alpha}{2}} e_{22}, \\ \hat{\lambda}_{21} \mp z_{\frac{\alpha}{2}} e_{33},\ \hat{\lambda}_{22} \mp z_{\frac{\alpha}{2}} e_{44}, \end{cases} \tag{28}$$

where the values e_{ii}, $i = 1, .., 5$ are the diagonals of the κ_0^{-1}. Also, the value z has standard normal distribution and the corresponding significant level is equal to γ. Equation (28) has shown that, lower bound of the asymptotic confidence intervals may be taken negative value. Therefore, I consider the delta method with log-transformation to avoid the negative cases as follows:

The log-transformation of of the model parameters $\Theta = \{\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \gamma\}$ are defined as $\log\Theta_i$, $i = 1, 2, .., 6$. Under normal property of the pivotal $\eta = \frac{\log\Theta_i - \log\hat{\Theta}_i}{\mathrm{Var}(\log\{\hat{\Theta}_i\})}$ with mean 0 and variance 1. Hence, the approximate $(1-\alpha)100\%$ confidence interval of the model parameters Θ is defined as

$$\left(\frac{\hat{\Theta}_i}{\exp\left(z_{\frac{\alpha}{2}}\sqrt{\mathrm{Var}(\log\hat{\Theta}_i)}\right)}, \hat{\Theta}_i \exp\left(z_{\frac{\alpha}{2}}\sqrt{\mathrm{Var}(\log\hat{\Theta}_i)}\right) \right), \tag{29}$$

where $\mathrm{Var}(\log\hat{\Theta}_i)=\frac{\mathrm{Var}(\hat{\Theta}_i)}{\hat{\Theta}_i}$ and $i = 1, 2, .., 5$, see [36].

7. BOOTSTRAP CONFIDENCE INTERVALS

Resembling methods “Bootstrap” for parameters estimation of a life populations are commonly used not only for estimated confidence intervals, but also in estimation the calibrate hypothesis tests or bias and variance of an estimator. In this section, the parametric bootstrap technique is adopted to formulate confidence interval of the parameters of the model, for more detail about parametric bootstrap techniques see, [37]. The following algorithms describe bootstrap-p and bootstrap-t confidence intervals as follows:

1- For given competing risks type II censoring sample $((w_1, \delta_1), (w_2, \delta_2), ..., (w_r, \delta_r), t < (w_{r+1}, \delta_{r+1}), ..., (w_m, \delta_m))$ the value of ML estimate is $\hat{\Theta} = \{\hat{\lambda}_{11}, \hat{\lambda}_{12}, \hat{\lambda}_{21}, \hat{\lambda}_{22}, \hat{\gamma}\}$.

2- Based on $\hat{\lambda}_{j1}$ and $\hat{\lambda}_{j2}$ generated two type II censoring sample from distribution given by Equation (9), $j = 1, 2$. The competing risks type II censoring sample is obtained minimum of two samples.

3- Applied the transformation in Equation (12) the bootstrap competing risks type II censoring sample is obtained as $((w_1^*, \delta_1^*), (w_2^*, \delta_2^*), ..., (w_r^*, \delta_r^*), t < (w_{r+1}^*, \delta_{r+1}^*), ..., (w_m^*, \delta_m^*))$.

4- Based on bootstrap competing risks type II censoring sample compute the bootstrap sample estimate $\hat{\Theta}^* = \{\hat{\lambda}_{11}^*, \hat{\lambda}_{12}^*, \hat{\lambda}_{21}^*, \hat{\lambda}_{22}^*, \hat{\gamma}^*\}$.

5- Steps from Step 2 to Step 4 are repeated N times.

6- The vector $\hat{\Theta}^{*(1)} = \{\hat{\lambda}_{11}^{*(i)}, \hat{\lambda}_{12}^{*(i)}, \hat{\lambda}_{21}^{*(i)}, \hat{\lambda}_{22}^{*(i)}, \hat{\gamma}^{*(i)}\}$, $i = 1, 2, ..., N$ put in ascending order as $\hat{\Theta}_{(i)}^* = \{\hat{\lambda}_{11(i)}^*, \hat{\lambda}_{12(i)}^*, \hat{\lambda}_{21(i)}^*, \hat{\lambda}_{22(i)}^*, \hat{\gamma}_{(i)}^*\}$, i =1, 2, ..., N.

Bootstrap-p confidence interval: Suppose that, the CDF of $\hat{\Theta}_k^*$ is defined by $Z(x) = P(\hat{\Theta}_k^* \leqslant x)$. The value $\hat{\Theta}_{k\text{boot}}^* = Z^{-1}(x)$ and $100(1-\alpha)\%$ is the approximate confidence interval of $\hat{\Theta}_k^*$, $k = 1$, 2, ..., 5 is formulated by:

$$\left[\hat{\Theta}_{k\text{boot}}^*(\frac{\alpha}{2}), \hat{\Theta}_{k\text{boot}}^*(1-\frac{\alpha}{2})\right]. \tag{30}$$

Bootstrap-t confidence interval Suppose, the order statistics values $\Omega_k^{*(1)} < \Omega_k^{*(2)} < ... < \Omega_k^{*(N)}$, is defined by

$$\Omega_k^{*(i)} = \frac{\hat{\Theta}_k^{*(i)} - \hat{\Theta}_k}{\sqrt{\text{var}\left(\hat{\Theta}_k^{*(i)}\right)}}, \; i = 1, \; 2, \; ..., \; N, \; k = 1, \; 2, \; ..., \; 5 \tag{31}$$

where $\hat{\Theta}_1 = \hat{\lambda}_{11}$, $\hat{\Theta}_2 = \hat{\lambda}_{12}$ and so.

Suppose, the CDF of $\hat{\Theta}_k^*$ is defined by $Z(x) = P(\Omega_k^* < x)$

$$\hat{\Theta}_{k\text{boot-}t} = \hat{\Theta}_k + \sqrt{\text{Var}(\hat{\Theta}_k)}Z^{-1}(x). \tag{32}$$

Hence, $100(1-\alpha)\%$ confidence interval of $\hat{\Theta}_k$ is given by:

$$\left(\hat{\Theta}_{k\text{boot-}t}(\frac{\alpha}{2}), \hat{\Theta}_{k\text{boot-}t}(1-\frac{\gamma}{2})\right).$$

8. SIMULATION STUDIES

In this section, the the studied model and the corresponding estimation methods are investigated under Monte Carlo simulation study. The results are tested with respect to several values of the model parameters. Also, I test the effect of change each of censoring scheme stress change time. The performance of the the estimators of the acceleration factor and scale parameters has been considered in regards to the mean square errors (MSEs) for different values of ζ. The approximate confidence intervals and two bootstrap confidence intervals (boot-p and boot-t) are tested with respect to average interval lengths (AIL) and coverage percentages (CP). A 1000 different samples are generated for each sample and checked whether the true value lays within the interval, then the length of the confidence interval is registered. Coverage percentage value is computed as the number of confidence intervals that covered the true values divided by 1000. However, the sum of the lengths for all intervals divided by 1000 is the estimated expected width. In this study, two values of the parameters are considered and accelerated factor, $\Theta = \{\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \gamma\} = \{1.5, 0.8, 1.2, 0.6, 2.0\}$ and the corresponding $(t = 0.5, 1.5)$, $\Theta = \{0.5, 0.8, 0.8, 0.2, 1.5\}$ with $(t = 1.0, 2.0)$. Without loss the generality the value of ζ is taken to be 0.4.

Results discussion: From the numerical results shown in TABLE 1 - TABLE 4, the following points can be noticed:

1- The results are logical.

2- By increasing the stress change times, the MSEs of the considered parameters decrease, for fixed values of the sample size.

3- For fixed values of the sample and failure time sizes, the type II censoring scheme serves well.

4- The approximate CIs and bootstrap-*t* CIs provide more accurate results than the bootstrap-p CIs, and that is due to the lengths of CIs and bootstrap-*t* are smaller than the lengths of bootstrap-p CIs, for different sample sizes, observed failures and schemes.

5- The accelerated factor estimate values performance under reduced value of *t* shows good results.

6- TABLE 1 - TABLE 4 indicate that estimation schemes are able to perform effectively for different choices of the model's parameter values.

9. CONCLUSION

The problem of analyzing time-to-failure of life units or individual is a common approach in competing risks models, however, under modern technologies, more time is needed for the failure occurrence specially in readable products. For this reason, to obtain sufficient information about life products in a small period of time, the ALTs problem is adopted. In this paper, the step-stress partially ALTs model is applied to analysis the type II censoring competing risks data obtained from mixed population. The mixture of two exponential distributions is the model is used for this work and this work can be extended to other mixture distributions.The results of simulation studies are

Table 1: The estimate value of mean and MSEs of $\Theta = (1.5, 0.8, 1.2, 0.6, 2.0)$.

t		MLE									
	(n, m)	AVG					MSE				
		λ_{11}	λ_{12}	λ_{21}	λ_{22}	γ	λ_{11}	λ_{12}	λ_{21}	λ_{22}	γ
0.5	(25,15)	1.854	1.123	1.452	0.874	2.542	0.321	0.245	0.331	0.210	0.452
	(25,25)	1.801	1.091	1.411	0.824	2.507	0.282	0.201	0.287	0.191	0.415
	(40,25)	1.791	1.094	1.415	0.821	2.502	0.279	0.197	0.282	0.194	0.407
	(40,35)	1.732	1.041	1.361	0.784	2.440	0.240	0.161	0.241	0.159	0.471
	(60,40)	1.690	1.002	1.301	0.719	2.391	0.208	0.142	0.217	0.118	0.438
	(60,50)	1.601	0.987	1.150	0.640	2.282	0.184	0.100	0.169	0.081	0.394
	(70,70)	1.578	0.894	1.201	0.615	2.174	0.101	0.071	0.132	0.044	0.351
1.5	(25,15)	1.814	1.084	1.415	0.823	2.500	0.301	0.218	0.307	0.194	0.433
	(25,25)	1.784	1.061	1.362	0.791	2.451	0.258	0.175	0.259	0.174	0.400
	(40,25)	1.762	1.044	1.365	0.789	2.462	0.254	0.175	0.257	0.169	0.389
	(40,35)	1.681	1.002	1.313	0.727	2.403	0.217	0.131	0.212	0.135	0.454
	(60,40)	1.654	0.978	1.241	0.694	2.359	0.192	0.117	0.204	0.100	0.424
	(60,50)	1.544	0.928	1.104	0.596	2.247	0.114	0.060	0.119	0.042	0.345
	(70,70)	1.541	0.852	1.154	0.596	2.124	0.087	0.043	0.114	0.025	0.328

given in TABLE 1 - TABLE 4 show the results are logical and the estimation methods behave well for different values of the model's parameters. This work can also, be extended to other censoring schemes and populations. also, the results can be applied to more than two causes of failure and others accelerated types.

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Table 2: The estimate value AIL and CP of Θ=(1.5, 0.8, 1.2, 0.6, 2.0).

t			MLE									
	(n,m)	CS	AIL					CP				
			λ_{11}	λ_{12}	λ_{21}	λ_{22}	γ	λ_{11}	λ_{12}	λ_{21}	λ_{22}	γ
0.5	(25,15)	ML	3.567	2.147	2.546	1.425	5.124	0.88	0.89	0.87	0.90	0.90
		Boot-p	3.687	2.251	2.664	1.547	5.288	0.88	0.89	0.89	0.91	0.89
		Boot-t	3.412	2.041	2.444	1.325	5.019	0.88	0.90	0.91	0.89	0.90
	(25,25)	ML	3.515	2.094	2.499	1.375	5.041	0.90	0.89	0.91	0.91	0.90
		Boot-p	3.619	2.204	2.607	1.491	5.227	0.91	0.92	0.91	0.90	0.92
		Boot-t	3.370	2.001	2.392	1.282	4.954	0.92	0.93	0.91	0.92	0.93
	(40,25)	ML	3.519	2.081	2.492	1.369	5.054	0.91	0.90	0.91	0.91	0.93
		Boot-p	3.612	2.201	2.614	1.487	5.215	0.91	0.94	0.91	0.93	0.92
		Boot-t	3.366	1.998	2.387	1.279	4.942	0.92	0.94	0.94	0.92	0.95
	(40,35)	ML	3.410	2.004	2.399	1.281	4.954	0.91	0.92	0.91	0.92	0.91
		Boot-p	3.511	2.114	2.524	1.403	5.124	0.91	0.94	0.93	0.95	0.91
		Boot-t	3.287	1.905	2.301	1.192	4.888	0.94	0.94	0.94	0.92	0.92
	(60,40)	ML	3.301	1.924	2.315	1.192	4.850	0.93	0.92	0.91	0.92	0.90
		Boot-p	3.412	2.025	2.424	1.327	5.018	0.92	0.94	0.92	0.95	0.94
		Boot-t	3.168	1.824	2.222	1.101	4.802	0.92	0.94	0.92	0.92	0.93
	(60,50)	ML	3.230	1.871	2.264	1.119	4.780	0.91	0.92	0.92	0.92	0.93
		Boot-p	3.351	1.981	2.354	1.262	5.018	0.90	0.93	0.92	0.94	0.91
		Boot-t	3.100	1.735	2.151	1.014	4.725	0.92	0.95	0.92	0.95	0.93
	(70,70)	ML	3.091	1.690	2.114	1.001	4.620	0.93	0.92	0.91	0.92	0.95
		Boot-p	3.214	1.874	2.194	1.114	4.871	0.94	0.93	0.95	0.94	0.93
		Boot-t	3.000	1.620	2.014	0.984	4.660	0.94	0.94	0.92	0.94	0.93
1.5	(25,15)	ML	3.545	2.131	2.519	1.402	5.101	0.90	0.89	0.82	0.91	0.91
		Boot-p	3.662	2.224	2.629	1.518	5.251	0.90	0.90	0.89	0.91	0.92
		Boot-t	3.395	2.017	2.418	1.299	5.000	0.90	0.91	0.92	0.89	0.90
	(25,25)	ML	3.470	2.066	2.470	1.325	4.982	0.90	0.89	0.91	0.91	0.93
		Boot-p	3.600	2.181	2.691	1.455	5.201	0.91	0.93	0.91	0.92	0.92
		Boot-t	3.352	1.987	2.376	1.261	4.925	0.92	0.92	0.93	0.92	0.91
	(40,25)	ML	3.502	2.071	2.459	1.344	5.018	0.91	0.92	0.93	0.92	0.91
		Boot-p	3.587	2.174	2.600	1.461	5.185	0.91	0.94	0.91	0.93	0.92
		Boot-t	3.339	1.974	2.364	1.254	4.919	0.92	0.93	0.94	0.94	0.95
	(40,35)	ML	3.387	1.974	2.372	1.254	4.918	0.92	0.95	0.94	0.92	0.94
		Boot-p	3.500	2.097	2.501	1.375	5.103	0.91	0.94	0.93	0.95	0.91
		Boot-t	3.269	1.879	2.282	1.169	4.851	0.94	0.94	0.94	0.92	0.92
	(60,40)	ML	3.269	1.901	2.292	1.171	4.824	0.93	0.92	0.91	0.92	0.90
		Boot-p	3.387	2.008	2.400	1.301	4.987	0.92	0.94	0.92	0.95	0.94
		Boot-t	3.144	1.798	2.200	1.087	4.759	0.92	0.94	0.92	0.92	0.93
	(60,50)	ML	3.204	1.839	2.225	1.100	4.731	0.91	0.92	0.92	0.92	0.93
		Boot-p	3.314	1.949	2.324	1.241	5.003	0.90	0.93	0.92	0.94	0.91
		Boot-t	3.71	1.714	2.134	0.989	4.725	0.92	0.95	0.92	0.95	0.93
	(70,70)	ML	3.066	1.672	2.092	0.997	4.602	0.93	0.92	0.91	0.92	0.95
		Boot-p	3.197	1.847	2.169	1.082	4.805	0.94	0.93	0.95	0.94	0.93
		Boot-t	2.974	1.602	1.984	0.971	4.635	0.94	0.94	0.92	0.94	0.93

Table 3: The estimate value of mean and MSEs of $\Theta = (0.5, 0.8, 0.8, 0.2, 1.5)$.

t		MLE									
	(n,m)	AVG					MSE				
		λ_{11}	λ_{12}	λ_{21}	λ_{22}	γ	λ_{11}	λ_{12}	λ_{21}	λ_{22}	γ
1.0	(25,15)	0.741	0.954	1.124	0.354	1.897	0.147	0.200	0.184	0.078	0.364
	(25,25)	0.709	0.925	1.092	0.312	1.871	0.125	0.181	0.168	0.069	0.349
	(40,25)	0.711	0.921	1.078	0.302	1.865	0.127	0.180	0.162	0.063	0.341
	(40,35)	0.674	0.881	1.039	0.266	1.831	0.101	0.161	0.129	0.044	0.319
	(60,40)	0.637	0.847	1.007	0.224	1.800	0.089	0.132	0.108	0.029	0.289
	(60,50)	0.612	0.818	0.987	0.198	1.761	0.066	0.108	0.087	0.022	0.256
	(70,70)	0.564	0.747	0.952	0.166	1.742	0.049	0.089	0.081	0.018	0.231
1.5	(25,15)	0.724	0.932	1.107	0.337	1.869	0.132	0.185	0.171	0.070	0.348
	(25,25)	0.689	0.902	1.071	0.280	1.855	0.114	0.171	0.160	0.054	0.334
	(40,25)	0.701	0.907	1.071	0.289	1.852	0.120	0.169	0.156	0.057	0.341
	(40,35)	0.659	0.874	1.031	0.251	1.818	0.092	0.147	0.118	0.040	0.308
	(60,40)	0.619	0.832	1.000	0.202	1.789	0.080	0.119	0.101	0.003	0.271
	(60,50)	0.600	0.807	0.976	0.191	1.747	0.054	0.100	0.078	0.020	0.245
	(70,70)	0.549	0.724	0.921	0.148	1.727	0.024	0.078	0.066	0.012	0.222

References

[1] Zhang X, Yang J, Kong X. A New Statistical Inference Method for Multi-Stress Accelerated Life Testing Based on Random Variable Transformation. Appl Math Modell. 2021;100:379-393.

[2] Louzada-Neto F, Mazucheli J, Achcar JA. Mixture Hazard Models for Lifetime Data. Biom J. 2002;44:3-14.

[3] Ishwaran H, James LF, Sun J. Bayesian Model Selection in Finite Mixtures by Marginal Density Decompositions. J Am Stat Assoc. 2001;96:1316-1332.

[4] Baey C, Cournède PH, Kuhn E. Asymptotic Distribution of Likelihood Ratio Test Statistics for Variance Components in Nonlinear Mixed Effects Models. Comp Stat Data Anal. 2019;135:107-122.

[5] Eberly LE, Casella G. Estimating Bayesian Credible Intervals. Journal of Statistical Planning and Inference. 2003;112:115-132.

[6] Crambes C, Delsol L, Laksaci A. Robust Nonparametric Estimation for Functional Data. J Nonparametric Stat. 2008;20:573-598.

[7] Rajarshi S, Rajarshi MB. Bathtub Distributions: A Review. Commun Stat Theor Methods. 1988;17:2597-2621.

[8] Holland PW, Laskey KB, Leinhardt S. Stochastic Blockmodels: First Steps. Soc Netw. 1983;5:109-137.

[9] Aitkin M. Likelihood and Bayesian Analysis of Mixtures. Stat Modell. 2001;1:287-304.

Table 4: The estimate value AIL and CP of $\Theta = (0.5, 0.8, 0.8, 0.2, 1.5)$.

t			MLE									
	(n,m)	CS	AIL					CP				
			λ_{11}	λ_{12}	λ_{21}	λ_{22}	γ	λ_{11}	λ_{12}	λ_{21}	λ_{22}	γ
1.0	(25,15)	ML	1.424	1.827	1.679	0.542	3.245	0.88	0.89	0.88	0.90	0.88
		Boot-p	1.687	1.998	1.874	0.745	3.369	0.89	0.90	0.89	0.90	0.90
		Boot-t	1.332	1.645	1.547	0.487	3.114	0.90	0.91	0.91	0.90	0.92
	(25,25)	ML	1.384	1.788	1.625	0.492	3.200	0.90	0.89	0.90	0.91	0.89
		Boot-p	1.633	1.955	1.831	0.701	3.314	0.89	0.92	0.91	0.91	0.90
		Boot-t	1.269	1.600	1.491	0.424	3.066	0.94	0.93	0.91	0.90	0.93
	(40,25)	ML	1.381	1.782	1.627	0.487	3.194	0.90	0.89	0.90	0.93	0.89
		Boot-p	1.635	1.949	1.828	0.693	3.305	0.90	0.92	0.91	0.91	0.90
		Boot-t	1.262	1.597	1.484	0.418	3.059	0.94	0.93	0.93	0.94	0.91
	(40,35)	ML	1.335	1.751	1.587	0.452	3.158	0.92	0.89	0.91	0.93	0.91
		Boot-p	1.600	1.903	1.777	0.659	3.275	0.92	0.92	0.95	0.91	0.93
		Boot-t	1.215	1.562	1.448	0.372	3.024	0.92	0.93	0.93	0.92	0.93
	(60,40)	ML	1.304	1.713	1.535	0.414	3.115	0.93	0.92	0.91	0.93	0.92
		Boot-p	1.564	1.801	1.744	0.622	3.241	0.92	0.92	0.91	0.92	0.93
		Boot-t	1.184	1.527	1.415	0.341	3.001	0.93	0.93	0.93	0.94	0.94
	(60,50)	ML	1.267	1.675	1.511	0.378	3.075	0.92	0.93	0.91	0.94	0.92
		Boot-p	1.532	1.777	1.711	0.578	3.213	0.92	0.92	0.95	0.92	0.95
		Boot-t	1.155	1.645	1.391	0.314	2.941	0.93	0.96	0.93	0.94	0.91
	(70,70)	ML	1.201	1.615	1.468	0.378	3.075	0.93	0.93	0.92	0.94	0.90
		Boot-p	1.469	1.722	1.650	0.524	3.157	0.92	0.92	0.95	0.94	0.94
		Boot-t	1.042	1.531	1.341	0.241	2.903	0.93	0.94	0.93	0.94	0.95
2.0	(25,15)	ML	1.411	1.808	1.664	0.532	3.230	0.88	0.87	0.89	0.90	0.89
		Boot-p	1.666	1.982	1.861	0.735	3.352	0.88	0.90	0.89	0.88	0.90
		Boot-t	1.314	1.635	1.541	0.471	3.100	0.92	0.92	0.91	0.91	0.92
	(25,25)	ML	1.377	1.781	1.611	0.481	3.187	0.90	0.91	0.90	0.92	0.92
		Boot-p	1.619	1.932	1.814	0.699	3.300	0.91	0.92	0.91	0.93	0.92
		Boot-t	1.251	1.571	1.469	0.407	3.042	0.92	0.93	0.91	0.90	0.93
	(40,25)	ML	1.371	1.765	1.611	0.472	3.181	0.93	0.89	0.90	0.93	0.91
		Boot-p	1.612	1.921	1.809	0.681	3.305	0.92	0.94	0.91	0.93	0.90
		Boot-t	1.248	1.581	1.471	0.403	3.044	0.94	0.93	0.95	0.94	0.93
	(40,35)	ML	1.321	1.744	1.580	0.439	3.141	0.93	0.92	0.91	0.92	0.95
		Boot-p	1.591	1.887	1.770	0.643	3.261	0.92	0.91	0.91	0.93	0.94
		Boot-t	1.202	1.548	1.441	0.364	3.011	0.92	0.94	0.93	0.92	0.94
	(60,40)	ML	1.291	1.702	1.518	0.397	3.107	0.94	0.90	0.91	0.92	0.92
		Boot-p	1.551	1.787	1.728	0.604	3.222	0.92	0.92	0.91	0.92	0.92
		Boot-t	1.178	1.508	1.404	0.325	2.974	0.94	0.93	0.96	0.94	0.96
	(60,50)	ML	1.244	1.670	1.500	0.359	3.061	0.92	0.93	0.94	0.94	0.92
		Boot-p	1.511	1.758	1.700	0.565	3.202	0.92	0.93	0.92	0.92	0.93
		Boot-t	1.139	1.624	1.379	0.305	2.924	0.93	0.90	0.93	0.94	0.93
	(70,70)	ML	1.189	1.601	1.448	0.369	3.061	0.93	0.91	0.93	0.94	0.92
		Boot-p	1.454	1.708	1.632	0.502	3.144	0.92	0.92	0.95	0.93	0.94
		Boot-t	1.028	1.500	1.322	0.218	2.887	0.93	0.93	0.93	0.94	0.94

[10] Nelson WB. Accelerated Testing: Statistical Models Test Plans and Data Analysis. John Wiley Sons. 2009.

[11] Kim CM, Bai DS. Analyses of Accelerated Life Test Data Under Two Failure Modes. Int J Reliab Qual Saf Eng. 2002;09:111-125.

[12] Chen N, Tang Y, Ye ZS. Robust Quantile Analysis for Accelerated Life Test Data. IEEE Trans Reliab. 2015;65:901-913.

[13] Xu A, Basu S, Tang Y. A Full Bayesian Approach for Masked Data in Step-Stress Accelerated Life Testing. IEEE Trans Reliab. 2014;63:798-806.

[14] Li Y, Pan G, Li Q, Wang C, Hu X. A Novel Accelerated Life-Test Method Under Thermal Cyclic Loadings for Electronic Devices Considering Multiple Failure Mechanisms. Microelectron Reliab. 2020;114:113766.

[15] Limnios N, Nikulin M. Recent Advances in Reliability Theory: Methodology Practice and Inference. 2012.

[16] Orbe J, Ferreira E, Núnez-Anton V. Comparing Proportional Hazards and Accelerated' Failure Time Models for Survival Analysis. Stat Med. 2002;21:3493-3510.

[17] Hauschild M, Jeswiet J, Alting L. From Life Cycle Assessment to Sustainable Production: Status and Perspectives. CIRP Ann. 2005;54:1-21.

[18] Dai H, Wei X, Sun Z, Wang J, Gu W. Online Cell Soc Estimation of LI-Ion Battery Packs Using a Dual Time-Scale Kalman Filtering for EV Applications. Appl Energy. 2012;95:227-237.

[19] Thomas EV, Bloom I, Christophersen JP, Battaglia VS. Statistical Methodology for Predicting Thex Life of Lithium-Ion Cells via Accelerated Degradation Testing. J Power Sources. 2008;184:312-317.

[20] Wong WC, Newman J. Monte Carlo Simulation of the Open-Circuit Potential and the Entropy of Reaction in Lithium Manganese Oxide. J Electrochem Soc. 2002;149:A493.

[21] Shi J. Intelligent Lifetime Test and Evaluation System for Tube Fuse. In: Second WRI Global Congress on Intelligent Systems. IEEE PUBLICATIONS. 2010;1:412-415.

[22] Zhongyi C, Yunxiang C, Zezhou W, Yuhui W. Lifetime Assessment Method Based on Step-Stress Accelerated Lifetime Test. In: Prognostics and System Health Management Conference (PHM-Harbin). IEEE PUBLICATIONS. 2017:1-5.

[23] Wang L, Pan R, Li X, Jiang T. A Bayesian Reliability Evaluation Method With Integrated Accelerated Degradation Testing and Field Information. Reliab Eng Syst Saf. 2013;112:38-47.

[24] Wang HW, Xu TX, Zhao JZ. Residual Life Prediction Method Fusing Accelerated Degradation and Field Degradation Data. Acta Aeronaut ET Astronaut Sin. 2014;35:3350-3357.

[25] Orelien JG, Edwards LJ. Fixed-Effect Variable Selection in Linear Mixed Models Using R2 Statistics. Comp Stat Data Anal. 2008;52:1896-1907.

[26] Antonio K, Beirlant J. Actuarial Statistics With Generalized Linear Mixed Models. Ins Math Econ. 2007;40:58-76.

[27] Pan R. A Bayes Approach to Reliability Prediction Utilizing Data From Accelerated Life Tests and Field Failure Observations. Qual Reliab Eng Int. 2009;25:229-240.

[28] Xiang-po Z, Jian-zhong S, Xun C, Chun-hua Z, Ya-shun W. Statistical Inference of Accelerated Life Test With Competing Failures Based on Three-Parameter Weibull Distribution. Acta Armamentarii. 2013;34:1603.

[29] McSorley EO, Lu JC, Li CS. Performance of Parameter-Estimates in Step-Stress Accelerated Life-Tests With Various Sample-Sizes. IEEE Trans Reliab. 2022;51:271-277.

[30] Cox DR. The Analysis of Exponentially Distributed Lifetimes With Two Types of Failures. J R Stat Soc. 1959;21:411-421.

[31] Soliman Tahani AA. A. Abushal, G.A. Abd-Elmougod. Inference of Partially Observed Causes for Failure of Lomax Competing Risks Model Under Type-II Generalized Hybrid Censoring Scheme. Alex Eng J. 2021;61:5427-5439.

[32] Farghal AW, Badr SK, Abu-Zinadah H, Abd-Elmougod GA. Analysis of Generalized Inverted Exponential Competing Risks Model in Presence of Partially Observed Failure Modes. Alex Eng J. 2023;78:74-87.

[33] Al-Essa LA, Soliman AA, Abd-Elmougod GA, Alshanbari HM. Comparative Study With Applications for Gompertz Models Under Competing Risks and Generalized Hybrid Censoring Schemes. Aximos. 2023;12:322.

[34] Al-Essa LA, Soliman AA, Abd-Elmougod GA, Alshanbari HM. Statistical Inference of Joint Competing Risks Models From Comparative Bathtub Shape Distributions With Hybrid Censoring. Alex Eng J. 2024;86:9-22.

[35] Arnold BC, Balakrishnan N, Nagaraja HN. A First Course in Order Statistics. SIAM.2008.

[36] Shih JH, Emura T. Likelihood-Based Inference for Bivariate Latent Failure Time Models With Competing Risks Under the Generalized Fgm Copula. Comput Stat. 2018;33:1293-1323.

[37] DiCiccio TJ, Efron B. Bootstrap Confidence Intervals. Stat Sci. 1996;11:189-228.