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Representation of a Crisp Set as a Pair of Dual Fuzzy Sets

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Abstract

Expert knowledge representations often fail to determine compatibility levels on all objects, and these levels are represented for a certain sampling of universe. The samplings for the fuzzy terms of the linguistic variable, whose compatibility functions are aggregated according to a certain problem, may also be different. In such a case, neither L.A. Zadeh's analysis of fuzzy sets and even the dual forms of developing today R.R. Yager's q-rung orthopair fuzzy sets cannot provide the necessary aggregations. This fact, as a given, can be considered as a source of new types of information, in order to obtain different levels of compatibility

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according to Zadeh, presented throughout the universe. This source of information can be represented as a pair $\langle A, f_A \rangle$, where there is some crisp subset of the universe A that determines the sampling of objects from the universe, and a function f_A determines the compatibility levels of the elements of that sampling. It is a notion of split fuzzy set, constructed in this article, that allows for the semantic representation and aggregation of such information. This notion is again and again based on the notion of Zadeh fuzzy set. In particular, the operation of splitting a crisp subset into dual fuzzy sets is introduced.

Definitions of set operations on split dual fuzzy-sets are presented in the paper. The proofs are also presented that follow naturally from definitions and previous results.An example of MADM is presented for illustration of the application of splitting operation.

Keywords: Fuzzy sets, Set splitting, Duality of imperfect information, Uncertainty and imprecision of imperfect information, Q-Rungorthopair fuzzy sets, Lattices.

1. INTRODUCTION

In the study of complex events analysis and synthesis problems, the use of L.A. Zadeh's theory of fuzzy sets [1] has the particular importance today, when the problems of semantic representation of expert qualitative information are quite acute due to the complicated nature of the objects under study. Existing approaches to measuring the degrees of compatibility precision of studying objects are no longer satisfactory to today's researchers. This is why the two sides of imprecision- the levels of object compatibility and incompatibility - are becoming more and more independent in new research [2–20]. This independence is due to the dual representation of evaluation. Duality is becoming an important element in the presentation of incomplete information today, and study on the imprecisions and uncertainties of modeling complex events deserves a great attention. The most common direction of these issues today is to represent the dual nature of information evaluation in some independent degrees of belonging and non-belonging. This idea first came from Atanasov [2].

The intuitionistic fuzzy sets (IFS) theory by Atanasov [2] represents a new extension of Zadeh's fuzzy sets (FS) theory [1]. Because to each element of IFS, as Intuitionistic fuzzy number (IFN) (μ, ν) is assigned a membership degree (μ), a non-membership degree (v) and a hesitancy degree $(1 - \mu, 1 - \nu)$, IFS is much capable to deal with vagueness than FS. IFS theory was extensively utilized in various problems of different areas [1, 2]. Definitions of main arithmetic operations on IFN are given in [2, 3]. The IFS theory is found to be intensively applicable in decision-making direction of research. After consideration of the huge number of existing materials, the authors of [5] presented a scientometric review on IFS studies. At the same time the IFN (μ, ν) has a serious constraint - the sum of membership and non-membership degrees must be or less than 1. Nevertheless, it may happen that a DM provides such data for certain attribute that the aforementioned sum is greater than 1 $(\mu + \nu > 1)$. To cope with such case Yager [6, 7] introduced the concept of the Pythagorean fuzzy set (PFS) as a generalization of IFS, where a Pythagorean fuzzy number (PFN) (μ, ν) has a weaker constraint - the sum of squared degrees of membership and non-membership satisfies the inequality $\mu^2 + \nu^2 \leq 1$. But in many expert orthopair assessments, neither PFNs nor IFNs can describe fully intellectual activity, because the assessment psychology of a decision maker (DM)is too intricate for hard decision-making, and the attribute's information is still problematic to express with PFNs or IFNs. This problem was solved by Yager again [8, 9].

He introduced the notion of a q-rung orthopair fuzzy set (q-ROFS), where $q \geq 1$, and the sum of the qth power of the degrees of membership and non-membership cannot exceed 1. For a q-rung orthopair fuzzy number (q-ROFN) we have ($\mu^q + \nu^q \leq 1$). The fundamentals of arithmetic operations on such numbers are presented in [10, 12]. Obviously, the q-ROFSs are generalization of IFSs and PFSs. The IFSs and PFSs represent the particular cases of the q-ROFSs for $q = 1$ and $q = 2$. Thus, q-ROFNs appear to be more suitable and capable for expressing DM's assessment information. Study on Aggregations of experts' q-rung orthopair fuzzy evaluations are actively developed in different works of authors of this paper in multi-criteria decision-making problems [10–22]. A completely different approach to dual representation of a fuzzy set is developed in [23]. In this paper, the concept of lower a-level sets of fuzzy sets is introduced, which is regarded as a dual concept of upper α-level sets of fuzzy sets. Authors introduces a new concept of dual fuzzy sets. Dual decomposition theorem is established. The dual arithmetic of fuzzy sets in R^1 is studied and established some interesting results based on the upper and lower α-level sets.

In practice, there are frequent cases when experts are unable to determine the levels of compatibility on all objects. In fact, these levels are represented by a certain sampling of the universe. Experts may make these samplings different. Samplings for the fuzzy terms of linguistic variables may also be different. But aggregations of such information are still needed, and the universe may not be fully represented at all. In such a case, neither the Zadeh fuzzy set analysis nor the dual forms presented here in the form of q-rung orthogonal fuzzy sets can provide the required aggregations.

Actually, it means the following. For any expert from certain universe $\Omega = \{\omega_1, \ldots, \omega_n\}$, a certain sampling of items $A = \{\omega_{i_1}, \ldots, \omega_{i_A}\}$ is available for evaluation. Suppose the compatibility levels generated by any expert are represented as some function $f_A(\omega) : A \to [0, 1]$, where the values are known only on the elements of the set $A \subset \Omega$. This data may be different for his/her other evaluations or for those of other experts. The new type of information source differs from that involved in determining the levels of compatibility according to Zadeh's point of view. In this case the source of information is presented by pairs $\langle A, f_A \rangle$. We are dealing with a source and data of a different nature. Namely the possibility of semantic representation of such information by the notion of split fuzzy set constructed in this article is offered, which is again and again based on Zadeh's concept of a fuzzy set. In particular, the operation of splitting a crisp subset into dual fuzzy sets is introduced. It is this dual, split fuzzy sets lattice that will create a unified environment for aggregating expert evaluations of different samplings.

The second section explains the operation of splitting a crisp set indicator into dual fuzzy sets. It also explains the notions of splitting representations for sets union, intersection, Cartesian product, and other operations indicators. The third section studies the lattice of split elements of the Boolean lattice of indicators *I*, where it is proved that the splitting lattice of all elements of this lattice $\widetilde{I}$ is the Braeuer lattice. A number of facts about the properties of this lattice are given. The fourth section explains the operation of splitting a crisp set, which is equivalent to the operation of splitting its own indicator. The main properties of this operation with some proofs are given. The concept of the generalized degree of the universe is explained, which is the lattice of the elements obtained by splitting all the subsets of the universe. It is proved that this lattice is the Brewer lattice. In order to study this lattice, the fifth section discusses some formulas for conditional pseudo-addition of a lattice element. The sixth section considers some properties of the operation of splitting of sets. The seventh section discusses the ideal of split elements of $\widetilde{I}$ and the ideal of their pseudo-additions. It is proved that this lattice is equivalent to the Boolean lattice *I*.

2. OPERATION OF SPLITTING OF AN INDICATOR

Consider the source of information discussed in the introduction for expert evaluations. Suppose that, for any expert from some universe $\Omega = \{\omega_1, \ldots, \omega_n\}$ a certain sampling of elements is available for evaluation. Suppose the compatibility levels generated by the expert are represented as a certain function $f_A(\omega) : A \to [0,1]$, where the values are known only to the elements of the set $A \subset \Omega$. This data may be different for his/her other evaluations as well for other experts. The new type of information source differs from that involved in determining the levels of compatibility according to Zadeh's point of view [1]. In this case the source of information is presented by pairs $\langle A, f_A \rangle$. Let $A \subset \Omega$ and $I_A \in \{0,1\}^{\Omega}$ be its indicator. Represent it in the following form

$$I_A(\omega) = f(\omega) I_A(\omega) + (1 - f(\omega)) I_A(\omega)\,, \;\; \omega \in \Omega\,, \tag{1}$$

where $f(\omega) : \Omega \to [0,1]$ is some continuation of the function $f_A(\omega) : A \to [0,1]$ on the universe Ω $(f(\omega) = f_A(\omega), \;\; \omega \in A)$.

Definition 1 *Let us call representation* (1) *a splitting of indicator I_A with respect to function f.*

Introduce notations:

$$\begin{aligned} I_{\widetilde{A}}(\omega) &\equiv f(\omega) I_A(\omega) \text{ and} \\ I_{\widetilde{A}^D}(\omega) &\equiv (1 - f(\omega)) I_A(\omega)\,. \end{aligned} \tag{2}$$

Indicators $I_{\widetilde{A}}\,,\; I_{\widetilde{A}^D} \in [0,1]^{\Omega}$ of two fuzzy subsets $\tilde{A},\, \tilde{A}^D \subset \Omega$ are called splitting of an indicator I_A of a subset $A \subset \Omega$ and

$$I_A = I_{\widetilde{A}} + I_{\widetilde{A}^D}. \tag{3}$$

Definition 2 *Indicators $I_{\widetilde{A}}\,,\; I_{\widetilde{A}^D} \in [0,1]^{\Omega}$ as well as fuzzy subsets $\tilde{A},\, \tilde{A}^D \subset \Omega$ are called dual, respectively.*

According to L. Zadeh [24] $I_{\widetilde{A}}$ is an indicator or membership function (compatibility function) of some fuzzy subset $\tilde{A}$. It is clear that splitting does not depend on the continuation of the function $f_A(\omega) : A \to [0,1]$. More exactly, the pair $\langle A, f_A \rangle$ induces a pair of splitting fuzzy sets $(\tilde{A}, \tilde{A}^D)$.

Example 1.1 *Let be given a set of digits $\Omega = \{0,1,2,3,4,5,6,7,8,9\}$. Let also be given the sampling of universe as some subset $A \subset \Omega$. For example, suppose that A is a set of odd digits - $A = \{1,3,5,7,9\}$ and let expert's evaluation only on this sampling is given by the function $f_A(\omega) : A \to [0,1]$, $f_A(\omega) = \frac{1}{\omega+1}$, $\omega \in A$. Let $f(\omega) : \Omega \to [0,1]$ be any continuation of the function $f_A(\omega)$ on Ω. Then the splitting of the I_A of the subset $A \subset \Omega$ on the universe Ω into two dual fuzzy sets (or their indicators (membership functions)) looks like this: $\tilde{A} = \{{}^{0}/_{0}, {}^{1}/_{(1/2)}, {}^{2}/_{0}, {}^{3}/_{(1/4)}, {}^{4}/_{0}, {}^{5}/_{(1/6)}, {}^{6}/_{0}, {}^{7}/_{(1/8)}, {}^{8}/_{0}, {}^{9}/_{(1/10)}\}$ and $\tilde{A}^D = \{{}^{0}/_{0}, {}^{1}/_{(1/2)}, {}^{2}/_{0}, {}^{3}/_{(3/4)}, {}^{4}/_{0}, {}^{5}/_{(5/6)}, {}^{6}/_{0}, {}^{7}/_{(7/8)}, {}^{8}/_{0}, {}^{9}/_{9/10}\}$.*

Practically, dual splitting fuzzy subsets $\left\langle \tilde{A},\; \tilde{A}^D \right\rangle$ are created as fuzzy subsets on the universe Ω. Practical interpretation looks like following: Sometimes for the description of some uncertain term of some linguistic variable on the elements of an universe we usually construct membership function. But for the extension of the information containing in the membership function to only on some elements of concrete crisp subset, we are splitting this set into dual split fuzzy subsets. So, the extended information is contained in dual fuzzy sets. Duality of this extension means that both fuzzy sets contain the same information, but codified in different ways.

It is to mention that, splitting dual fuzzy subsets $\widetilde{A}$, $\widetilde{A}^D$ on Ω are induced by the subset $A,\ A \subset \Omega$ and some function $f_A(\omega):\ A \to [0,1]$.

As mentioned earlier, the possibility of using the split operation can arise in many cases. Here is one case.Let now consider an example on application of splitting a set into dual fuzzy sets inmulti-attribute decision making (MADM).

Consider a MADM model with 5 attributes $S = \{s_1, s_2, \ldots, s_5\}$ and 3 alternatives $D = \{d_1, d_2, d_3\}$. Suppose that a decision-making matrix represents a matrix of normed ratings in [0,1], where some ratings are not given:

D/S	s_1	s_2	s_3	s_4	s_5
d_1	0.2	-	0.7	0.6	-
d_2	-	0.4	-	0.3	0.8
d_3	0.3	0.5	0.6	-	-

As can be seen from this matrix, for each alternative there are attributes for which the rating evaluations are not presented. Such unusual cases can arise in practice for many reasons. One is when the number of attributes is quite large due to the deep detailing of the task, and it is difficult for experts to make a rating assessment on all attributes. Such cases often arise when building recommendation models in collaborative filtering problems. These empty elements need to be filled somehow. This problem can be successfully implemented with a machine learning approach, if, of course, there is a large amount of prehistoric data. Otherwise, when we do not have objective data and expert evaluations are of the sparse type, the splitting operation presented here can be a way out! We see that the alternative d_1 is evaluated on a subset of attributes $S_1 \equiv \{s_1, s_3, s_4\}$, the alternative d_2 is evaluated on a subset of attributes $S_2 \equiv \{s_2, s_4, s_5\}$, and the alternative d_3 is evaluated on a subset of attributes $S_3 \equiv \{s_1, s_2, s_3\}$. Let us split these sets into dual fuzzy sets. Then the decision matrix can be written as:

D/S	s_1	s_2	s_3	s_4	s_5
$\tilde{A}_{d_1}$	0.2	0.0	0.7	0.6	0.0
$\tilde{A}_{d_1}^D$	0.8	0.0	0.3	0.4	0.0
$\tilde{A}_{d_2}$	0.0	0.4	0.0	0.3	0.8
$\tilde{A}_{d_2}^D$	0.0	0.6	0.0	0.7	0.2
$\tilde{A}_{d_3}$	0.3	0.5	0.6	0.0	0.0
$\tilde{A}_{d_3}^D$	0.7	0.5	0.4	0.0	0.0

Therefore, the alternative $d_i\ \ i = 1, 2, 3$ is represented by dual split fuzzy subsets $\left\langle A_{d_i}, A_{d_i}^D \right\rangle$ on the whole universe of attributes $S = \{s_1, s_2, \ldots, s_5\}$.The creation of an aggregation instrument and the ways of constructing ranking relations can be developed in many directions, where the definitions and results presented in the following paragraphs on the operations of dual split sets will be used. Here is a simple solution. Combine the elements of split dual fuzzy sets into pairwise intuitionistic fuzzy numbers by a simple concatenation:

Here, of course, attention is drawn to the symbolic intuitionistic fuzzy number (0.0,0.0), whose attribution and non-attribution values are 0.0, which indicates the information that the evaluation is not done. As a matter of fact, if we gave a formula-quantitative value to ratings that are not evaluated by such representations, it is natural to replace it with zero intuitionistic fuzzy numerical rating - (0.0,1.0). Then the decision-making matrix takes the following form.

D/S	s_1	s_2	s_3	s_4	s_5
d_1	(0.2,0.8)	(0.0,1.0)	(0.7,0.3)	(0.6,0.4)	(0.0,1.0)
d_2	(0.0,1.0)	(0.4,0.6)	(0.0,1.0)	(0.7,0.3)	(0.8,0.2)
d_3	(0.3,0.7)	(0.5,0.5)	(0.6,0.4)	(0.0,1.0)	(0.0,1.0)

Suppose that the vector of attribute weights in this model is $W = \{w_1, w_2, \ldots, w_5\} = \{0.1,\ 0.2,\ 0.4,\ 0.1,\ 0.2\}$.

For ranking of alternatives let us use the intuitionistic fuzzy weighted averaging (IFWA) operator:

$$d_i \sim IFWA(a_1, \ldots, a_5).$$

For example, for d_1 we will have:

$$d_1 \sim 0.1 \otimes (0.2, 0.8) \oplus 0.2 \otimes (0.0, 1.0) \oplus 0.4 \otimes (0.7, 0.3) \oplus 0.1 \otimes (0.6, 0.4) \oplus 0.2 \otimes (0.0, 1.0)\,,$$

where $\oplus$ and $\otimes$ denote addition and multiplication operations on intuitionistic fuzzy numbers, respectively.

2.1 Splitting of an ordinary complement indicator I_{A^C}

Easy to see that for given splitting I_A the splitting I_{A^C} does not depend on $I_{\widetilde{A}}$, since $I_{A^C} I_A = 0$. Formally,

$$I_{A^C} = {I_{A^C}}^2 = (I_\Omega - I_{\widetilde{A}} - I_{\widetilde{A}^D}) I_{A^C} = I_{\neg\widetilde{A}} I_{A^C} + (1 - I_{\neg\widetilde{A}}) I_{A^C}\,, \tag{4}$$

where we introduced notation

$$I_{\neg\widetilde{A}} = 1 - I_{\widetilde{A}} = I_{\widetilde{A}^D} \vee I_{A^C}. \tag{5}$$

Therefore,

$$\begin{aligned} I_{\widetilde{A^C}} &= I_{\neg\widetilde{A}} I_{A^C} = (I_{\widetilde{A}^D} \vee I_{A^C}) I_{A^C} = I_{A^C}, \\ I_{\widetilde{A^C}^D} &= (1 - I_{\neg\widetilde{A}}) I_{A^C} = I_{\widetilde{A}} I_{A^C} = 0. \end{aligned} \tag{6}$$

2.2 Splitting of an indicator of intersection $I_{A\cap B}$

For any $f(\omega)\,,\;\; g(\omega)\in[0,1]^{\Omega}$ we can write splittings of I_A and I_B.

$$\begin{aligned} I_A(\omega) &= f(\omega)I_A(\omega)+(1-f(\omega))I_A(\omega)\,,\\ I_B(\omega) &= g(\omega)I_B(\omega)+(1-g(\omega))I_B(\omega)\,. \end{aligned} \tag{7}$$

Further,

$$\begin{aligned} I_{A\cap B}(\omega) &= I_A(\omega)\wedge I_B(\omega)\\ &= I_A(\omega)I_B(\omega)\\ &= f(\omega)g(\omega)I_A(\omega)I_B(\omega)+f(\omega)(1-g(\omega))I_A(\omega)I_B(\omega)\\ &\quad +(1-f(\omega))g(\omega)I_A(\omega)I_B(\omega)+(1-f(\omega))(1-g(\omega))I_A(\omega)I_B(\omega). \end{aligned} \tag{8}$$

If we require split indicators to meet the same condition as non-split indicators

$$I_{\widetilde{A\cap B}}\le I_{\widetilde{A}}\,,I_{\widetilde{B}}\;, \tag{9}$$

then from (8) we have two equivalentrepresentations:

$$I_{\widetilde{A\cap B}}=(I_{\widetilde{A}}\wedge I_{\widetilde{B}}) \tag{10}$$

and

$$I_{\widetilde{A\cap B}}=(I_{\widetilde{A}}\bullet I_{\widetilde{B}}). \tag{11}$$

Indeed, in the first case we have:

$$\begin{aligned} I_{A\cap B}(\omega) &= \begin{cases} [f(\omega)g(\omega)I_A\bullet I_B+f(\omega)(1-g(\omega))I_A\bullet I_B]+[(1-f(\omega))g(\omega)I_A\bullet I_B\\ \quad +(1-f(\omega))(1-g(\omega))I_A\bullet I_B],\;\;\forall\omega:f(\omega)\le g(\omega)\,;\\ [f(\omega)g(\omega)I_A\bullet I_B+(1-f(\omega))g(\omega)I_A\bullet I_B]\\ \quad +[f(\omega)(1-g(\omega))I_A\bullet I_B+(1-f(\omega))(1-g(\omega))I_A\bullet I_B]\,,\;\;\forall\omega:g(\omega)\le f(\omega); \end{cases}\\ &= \begin{cases} f(\omega)I_A(\omega)(I_A\bullet I_B)+(1-f(\omega))I_A(\omega)(I_A\bullet I_B)\,,\;\;\forall\omega:\;f(\omega)\le g(\omega);\\ g(\omega)I_B(\omega)(I_A\bullet I_B)+(1-g(\omega))I_B(\omega)(I_A\bullet I_B)\,,\;\;\forall\omega:\;g(\omega)\le f(\omega); \end{cases}\\ &= (I_{\widetilde{A}}(\omega)\wedge I_{\widetilde{B}}(\omega))(I_A\bullet I_B)+(I_{\widetilde{A}}(\omega)\wedge I_{\widetilde{B}}(\omega))^D(I_A\bullet I_B)\\ &= I_{\widetilde{A}}(\omega)\wedge I_{\widetilde{B}}(\omega)+(I_{\widetilde{A}}(\omega)\wedge I_{\widetilde{B}}(\omega))^D. \end{aligned} \tag{12}$$

In the second case, if we combine the terms of (8) like this:

$$I_{A\cap B}=(I_{\widetilde{A}}\bullet I_{\widetilde{B}})(I_A\bullet I_B)+[(I_{\widetilde{A}}\bullet I_{\widetilde{B}^D})+(I_{\widetilde{A}^D}\bullet I_{\widetilde{B}})+(I_{\widetilde{A}^D}\bullet I_{\widetilde{B}^D})]\,,$$

i.e., if we consider $(I_{\widetilde{A}}\bullet I_{\widetilde{B}})$ as the result of splitting $I_{A\cap B}$, then this splitting will also satisfy condition (9), i.e.

$$I_{A\cap B}=(I_{\widetilde{A}}\bullet I_{\widetilde{B}})I_{A\cap B}+(I_{\widetilde{A}}\bullet I_{\widetilde{B}})^D I_{A\cap B}. \tag{13}$$

In contrast with $(I_{\widetilde{A}}\wedge I_{\widetilde{B}})$, we call expression $(I_{\widetilde{A}}\bullet I_{\widetilde{B}})$ a "sequential" splitting of the indicator $I_{A\cap B}$. Indeed, consider $I_{A\cap B}$ and split I_A, we get

$$I_{A\cap B}=(I_{\widetilde{A}}+I_{\widetilde{A}^D})I_B=(I_{\widetilde{A}}\bullet I_B)I_{A\cap B}+(I_{\widetilde{A}^D}\bullet I_B)I_{A\cap B}.$$

Therefore, in this case

$$I_{\widetilde{A\cap B}} = (I_{\widetilde{A}} \bullet I_B) I_{A\cap B}.$$

Now split $I_{A\cap B}$ sequentially by help of function I_B, we get:

$$I_{\widetilde{A\cap B}} = I_{\widetilde{B}} \bullet I_{\widetilde{A\cap B}} + (1 - I_{\widetilde{B}}) I_{\widetilde{A\cap B}}.$$

Thus, in this case

$$I_{\widetilde{\widetilde{A\cap B}}} = I_{\widetilde{B}} \bullet I_{\widetilde{A\cap B}} = I_{\widetilde{B}} \bullet (I_{\widetilde{A}} \bullet I_B) \bullet I_{A\cap B} = I_{\widetilde{A}} \bullet I_{\widetilde{B}}. \tag{14}$$

2.3 Splitting of an indicator of union $I_{A\cup B}$

The splitting of an indicator $I_{A\cup B}$ can be constructed from the following relationships:

$$\begin{aligned} I_{A\cup B}(\omega) &= I_{A\cup(A^C\cap B)}(\omega) = I_A(\omega) + I_{A^C\cap B}(\omega) = I_A(\omega) + I_{A^C}(\omega) I_B(\omega) \\ &= f(\omega) I_A(\omega) + (1 - f(\omega)) I_A(\omega) + I_{A^C}(\omega)(g(\omega) I_B(\omega) + (1 - g(\omega)) I_B(\omega)) \\ &= f(\omega) I_A(\omega) + g(\omega) I_{A^C}(\omega) I_B(\omega) + (1 - f(\omega)) I_A(\omega) + (1 - g(\omega)) I_{A^C}(\omega) I_B(\omega). \end{aligned}$$

Analogously,

$$\begin{aligned} I_{A\cup B}(\omega) &= I_{B\cup(A\cap B^C)}(\omega) = I_B(\omega) + I_{A\cap B^C}(\omega) = I_B(\omega) + I_{B^C}(\omega) I_A(\omega) \\ &= g(\omega) I_B(\omega) + (1 - g(\omega)) I_B(\omega) + I_{B^C}(\omega)(f(\omega) I_A(\omega) + (1 - f(\omega)) I_A(\omega)) \\ &= g(\omega) I_B(\omega) + f(\omega) I_{B^C}(\omega) I_A(\omega) + (1 - g(\omega)) I_B(\omega) + (1 - f(\omega)) I_{B^C}(\omega) I_A(\omega)\,. \end{aligned}$$

Taking into account the property of idempotency of indicators we have:

$$\begin{aligned} I_{A\cup B}(\omega) &= \begin{cases} [f(\omega) I_A(\omega) + g(\omega) I_{A^C}(\omega) I_B(\omega)] I_{A\cup B}(\omega) \\ \quad +[(1 - f(\omega) I_A(\omega) + g(\omega) I_{A^C}(\omega) I_B(\omega))] I_{A\cup B}(\omega); \\ [g(\omega) I_B(\omega) + f(\omega) I_A(\omega) I_{B^C}(\omega)] I_{A\cup B}(\omega) \\ \quad +[(1 - g(\omega) I_B(\omega) + f(\omega) I_A(\omega) I_{B^C}(\omega))] I_{A\cup B}(\omega); \end{cases} \\ &= \begin{cases} [f(\omega) I_A(\omega) \vee g(\omega) I_{A^C}(\omega) I_B(\omega)] I_{A\cup B}(\omega) \\ \quad +[(1 - f(\omega) I_A(\omega) \vee g(\omega) I_{A^C}(\omega) I_B(\omega))] I_{A\cup B}(\omega); \\ [g(\omega) I_B(\omega) \vee f(\omega) I_A(\omega) I_{B^C}(\omega)] I_{A\cup B}(\omega) \\ \quad +[(1 - g(\omega) I_B(\omega) \vee f(\omega) I_A(\omega) I_{B^C}(\omega))] I_{A\cup B}(\omega)\,. \end{cases} \end{aligned} \tag{15}$$

If you require split indicators to meet the same condition as non-split indicators:

$$I_{\widetilde{A\cup B}} \geq I_{\widetilde{A}}\,,\, I_{\tilde{\tilde{B}}}, \tag{16}$$

i.e., if we consider that the choice of the law of point grouping should provide the order for split indicators

$$\begin{aligned} \max\left(f(\omega) I_A(\omega)\,,\quad g(\omega) I_B(\omega)\right) &\leq \left(f(\omega) I_A(\omega) \vee g(\omega) I_{A^C}(\omega) I_B(\omega)\right) \\ &\quad \vee \left(g(\omega) I_B(\omega) \vee f(\omega) I_A(\omega) I_{B^C}(\omega)\right) \\ &= \left(f(\omega) I_A(\omega) \vee f(\omega) I_A(\omega) I_{B^C}(\omega)\right) \\ &\quad \vee \left(g(\omega) I_B(\omega) \vee g(\omega) I_{A^C}(\omega) I_B(\omega)\right) \\ &= \left(f(\omega) I_A(\omega) \vee g(\omega) I_B(\omega)\right), \end{aligned} \tag{17}$$

then (15) can be rewritten as follows

$$\begin{aligned} I_{A\cup B}(\omega) &= (f(\omega)I_A(\omega)\vee g(\omega)I_B(\omega))I_{A\cup B}(\omega) + (1-(f(\omega)I_A(\omega)\vee g(\omega)I_B(\omega)))I_{A\cup B}(\omega) \\ &= (I_{\widetilde{A}}\vee I_{\widetilde{B}})(\omega)I_{A\cup B}(\omega) + (1-(I_{\widetilde{A}}\vee I_{\widetilde{B}}))(\omega)I_{A\cup B}(\omega), \end{aligned}$$

or

$$I_{\widetilde{A\cup B}} = I_{\widetilde{A}}\vee I_{\widetilde{B}}. \tag{18}$$

Now let us consider the procedure of "sequential" splitting. We have

$$I_{A\cup B} = I_A + I_B - I_{A\cap B}.$$

With "simultaneous" splitting I_A and I_B, we get:

$$I_{\widetilde{A\cup B}} = I_{\widetilde{A}} + I_{\widetilde{B}} - I_{\widetilde{A}\cap\widetilde{B}},$$

Since

$$I_{\widetilde{A}}\vee I_{\widetilde{B}} + I_{\widetilde{A}}\wedge I_{\widetilde{B}} = I_{\widetilde{A}} + I_{\widetilde{B}}\,,$$

then this implies that when the union indicator is "simultaneously" split

$$I_{\widetilde{A\cup B}} = I_{\widetilde{A}}\vee I_{\widetilde{B}} = I_{\widetilde{A}\cup\widetilde{B}}.$$

With "sequential" splitting, obviously,

$$I_{\widetilde{\widetilde{A\cup B}}} = I_{\widetilde{A}} + I_{\widetilde{B}} - I_{\widetilde{A}}\bullet I_{\widetilde{B}}. \tag{19}$$

2.4 Splitting of an Indicator of Cartesian Product $I_{A\times B}$

Let A , B be any subsets of universes Ω_1 and Ω_2, respectively. For ordinary subsets

$$I_{A\times B}(\omega_1\,,\omega_2) = I_{(A\times\Omega_2)\cap(\Omega_1\times B)}(\omega_1\,,\omega_2), \tag{20}$$

where $A\times B\subseteq\Omega_1\times\Omega_2$.

Let us split $I_A(\omega_1)$ and $I_B(\omega_2)$. Due to subsection 1.2 of this paper:

$$\begin{aligned} I_{\widetilde{A\times B}}(\omega_1\,,\omega_2) &= I_{\widetilde{(A\times\Omega_2)\cap(\Omega_1\times B)}}(\omega_1\,,\omega_2) \\ &= \begin{cases} I_{\widetilde{A\times\Omega_2}}(\omega_1\,,\omega_2)\wedge I_{\widetilde{\Omega_1\times B}}(\omega_1\,,\omega_2)\,, & (simultaneos\,splitting)\,, \\ I_{\widetilde{A\times\Omega_2}}(\omega_1\,,\omega_2)\bullet I_{\widetilde{\Omega_1\times B}}(\omega_1\,,\omega_2)\,, & (sequential\,splitting)\,. \end{cases} \end{aligned}$$

Since for crisp subsets we have

$$I_{A\times\Omega_2}(\omega_1\,,\omega_2) = I_A(\omega_1) \text{ and } I_{\Omega_1\times B}(\omega_1,\omega_2) = I_B(\omega_2),$$

then similar relationships occur for fuzzy subsets, i.e.

$$I_{\widetilde{A\times\Omega_2}}(\omega_1\,,\omega_2) = I_{\widetilde{A}}(\omega_1) \text{ and } I_{\widetilde{\Omega_1\times B}}(\omega_1\,,\omega_2) = I_{\widetilde{B}}(\omega_2).$$

Finally,

$$I_{\widetilde{A\times B}}(\omega_1,\omega_2) = \begin{cases} I_{\widetilde{A}}(\omega_1\,)\wedge I_{\widetilde{B}}(\omega_2)\;, & (simultaneos\,splitting) \\ I_{\widetilde{A}}(\omega_1\,)\bullet I_{\widetilde{B}}(\omega_2)\;, & (sequential\,splitting). \end{cases} \tag{21}$$

2.5 Decomposition of a Splitting Indicator $I_{\widetilde{A}}$

Let $\alpha \in [0,1]$. The indicator of the α-level of the split indicator will be called the non-split indicator

$$I_{A_\alpha}(\omega) = I_{\{\omega:\, \omega\subseteq\Omega\,,\, I_{\widetilde{A}}(\omega)\geq\alpha\}}\,, \tag{22}$$

A_α is a level set of the fuzzy set $\widetilde{A}$.

Further, let us call the elementary splitting of the α-level of the universal set Ω a fuzzy subset determined by the indicator:

$$I_{\widetilde{\Omega}_\alpha}(\omega) = \alpha\,,\ \ \forall\omega \in \Omega. \tag{23}$$

The following important property is valid:

$$\forall\,\alpha_1\,,\alpha_2\ \in [0,1]:\ \alpha_1 \geq \alpha_2 \Rightarrow I_{A_{\alpha_1}}(\omega) \leq I_{A_{\alpha_2}}(\omega),\ \ \omega \in \Omega. \tag{24}$$

Theorem 1 (Decomposition theorem) *Let $I_{\widetilde{A}}$ be an arbitrary split indicator and $\{I_{A_\alpha}\}$ be level sets, then*

$$I_{\widetilde{A}}(\omega) = \bigvee_{\alpha\in[0,1]}(I_{\widetilde{\Omega}_\alpha}(\omega) \wedge I_{A_\alpha}(\omega)). \tag{25}$$

Proof The proof follows directly from (22) and (23).

The decomposition theorem can be applied not only for analysis, but also for synthesis. If we consider some chain (meaning the natural ordering) of non-split indicators $\{I_\alpha\}$ and if (24) is fulfilled, then according to (25) we get a kind of split indicator. ■

3. Lattice of split elements of the Boolean lattice of indicators *i*

In any Boolean lattice *B*, the element a^C (complement of a) is the largest among those x that $a \wedge x = 0$. Generally, $a \wedge x \leq b$ if and only if $a \wedge x \wedge b^C = 0$,i.e. when $(a \wedge b^C) \wedge x = 0$, or $x \leq (a \wedge b^C)^C = a^C \vee b$. In addition, the element a^C has special properties:

$$a^C \vee a = E \text{ and } a^C \wedge a = 0. \tag{26}$$

If we take $i = ([0,1]^\Omega\,; \vee, \wedge)$ in the role of *B*, then the elements will be the indicators of subsets, and $E = I_\Omega$, $0 = I_\emptyset$. Consider all possible splitting of the elements of *i*. The set of all these split elements $\tilde{i} = ([0,1]^\Omega\,; \vee, \wedge)$, ordered in a natural manner, is a lattice.

Theorem 2 *The lattice $\tilde{i}$ is the Brewer lattice.*

See the proof in Appendix A.

Remark 1 *The relative pseudo-complement of $I_{\widetilde{A}}$ in $I_{\widetilde{B}}$, as well as the split complement $I_{\widetilde{A}^C}$, does not depend on the splitting of the indicator I_A.*

Theorem 3 *In the lattice $\tilde{i}$ we have the followingpropositions. If $\tilde{A}, \tilde{B} \in \tilde{i}$ then*

$$\begin{aligned}
&(i)\ if\ I_{\widetilde{A}} \le I_{\widetilde{B}}\,,\ \ then\ \ (I_\emptyset : I_{\widetilde{B}}) \le (I_\emptyset : I_{\widetilde{A}})\,;\\
&(ii)\ \ I_{\widetilde{A}} \le (I_\emptyset : (I_\emptyset : I_{\widetilde{A}}))\,;\\
&(iii)\ (\,I_\emptyset : I_{\widetilde{A}}) = (I_\emptyset : (I_\emptyset : (I_\emptyset : I_{\widetilde{A}})))\,;\\
&(iv)\ (\,I_\emptyset : (I_{\widetilde{A}} \vee I_{\widetilde{B}})) = (I_\emptyset : I_{\widetilde{A}}) \wedge (I_\emptyset : I_{\widetilde{B}})\,;\\
&(v)\ \ (\,I_\emptyset : (I_{\widetilde{A}} \wedge I_{\widetilde{B}})) = (I_\emptyset : I_{\widetilde{A}}) \vee (I_\emptyset : I_{\widetilde{B}})\,.
\end{aligned} \tag{27}$$

See the proof in Appendix A.

Theorem 4 *In the lattice $\tilde{i}$ we have the following propositions. If $\tilde{A}, \tilde{B}, \tilde{C} \in \tilde{i}$, then*

$$\begin{aligned}
&(i)\ \ (I_{\widetilde{A}} : I_{\widetilde{B}}) \wedge I_{\widetilde{A}} = I_{\widetilde{A}}\ ;\\
&(ii)\ \ (I_{\widetilde{A}} : I_{\widetilde{B}}) \wedge I_{\widetilde{B}} = I_{\widetilde{A}} \wedge I_{\widetilde{B}}\ ;\\
&(iii)\ \ ((I_{\widetilde{A}} \wedge I_{\widetilde{B}}) : I_{\widetilde{C}}) = (I_{\widetilde{A}} : I_{\widetilde{C}}) \wedge (I_{\widetilde{B}} : I_{\widetilde{C}})\ ;\\
&(iv)\ \ (I_{\widetilde{A}} : (I_{\widetilde{B}} \vee I_{\widetilde{C}})) = (I_{\widetilde{A}} : I_{\widetilde{B}}) \wedge (I_{\widetilde{A}} : I_{\widetilde{C}})\,.
\end{aligned} \tag{28}$$

See the proof in Appendix A.

4. SPLITTING OF A SET

Definition 3 *Splittingof a crisp set is equivalent to the splitting of the corresponding indicator and is formally represented as follows*

$$(I_A = I_{\widetilde{A}} + I_{\widetilde{A}^D} \rightleftarrows (A = \tilde{A} \oplus \tilde{A}^D) \tag{29}$$

where $\oplus$ is a splitting operation of a set.

As it was mentioned, from the Def.1 for the splitting of a crisp set A the building materials are an indicator I_A and some function $f_A(\omega) : A \to [0, 1]$. We interpret the split set on $\widetilde{A}$ into and its dual split set $\widetilde{A}^D$ as a fuzzy subset in the sense of Zadeh [24]. With this interpretation, $I_{\tilde{A}}$ is considered as an indicator (membership function) of a fuzzy split subset $\tilde{A}$.

By virtue of Def. 3, formulas (4), (9) and (10) we have the following relationships:

$$(I_{\widetilde{A \cap B}} = I_{\widetilde{A}} \wedge I_{\widetilde{B}}) \rightleftarrows (\widetilde{A \cap B} = \widetilde{A} \cap \widetilde{B}), \tag{30}$$

$$(I_{\widetilde{A \cup B}} = I_{\widetilde{A}} \vee I_{\widetilde{B}}) \rightleftarrows (\widetilde{A \cup B} = \widetilde{A} \cup \widetilde{B}), \tag{31}$$

$$(I_{\neg\widetilde{A}} = I_\Omega - I_{\widetilde{A}}) = (\neg\widetilde{A}). \tag{32}$$

In addition, as a definition, we also consider the relationship:

$$(I_{\widetilde{A}} \le I_{\widetilde{B}}) \rightleftarrows (\widetilde{A} \subseteq \widetilde{B}). \tag{33}$$

For split sets, the following laws are very easily verified:

1. Reflexivity:
$$\widetilde{A} \subseteq \widetilde{A}. \tag{34}$$
2. Anti-symmetricity:
$$(\widetilde{A} \subseteq \widetilde{B},\, \widetilde{B} \subseteq \widetilde{A}) \Rightarrow (\widetilde{A} = \widetilde{B}). \tag{35}$$
3. Transitivity:
$$(\widetilde{A} \subseteq \widetilde{B},\, \widetilde{B} \subseteq \widetilde{C}) \Rightarrow (\widetilde{A} \subseteq \widetilde{C}). \tag{36}$$
4. Idempotency:
$$\begin{gathered} \widetilde{A} \cap \widetilde{A} = \widetilde{A} \\ \widetilde{A} \cup \widetilde{A} = \widetilde{A}. \end{gathered} \tag{37}$$
5. Commutativity:
$$\begin{gathered} \widetilde{A} \cap \widetilde{B} = \widetilde{B} \cap \widetilde{A} \\ \widetilde{A} \cup \widetilde{B} = \widetilde{B} \cup \widetilde{A}. \end{gathered} \tag{38}$$
6. Associativity:
$$\begin{gathered} (\widetilde{A} \cap \widetilde{B}) \cap \widetilde{C} = \widetilde{A} \cap (\widetilde{B} \cap \widetilde{C}) \\ (\widetilde{A} \cup \widetilde{B}) \cup \widetilde{C} = \widetilde{A} \cup (\widetilde{B} \cup \widetilde{C}). \end{gathered} \tag{39}$$
7. Distributivity:
$$\begin{gathered} \widetilde{A} \cap (\widetilde{B} \cup \widetilde{C}) = (\widetilde{A} \cap \widetilde{B}) \cup (\widetilde{A} \cap \widetilde{C}) \\ \widetilde{A} \cup (\widetilde{B} \cap \widetilde{C}) = (\widetilde{A} \cup \widetilde{B}) \cap (\widetilde{A} \cup \widetilde{C}). \end{gathered} \tag{40}$$
8. Absorption laws:
$$\begin{gathered} \widetilde{A} \cap (\widetilde{A} \cup \widetilde{B}) = \widetilde{A} \\ \widetilde{A} \cup (\widetilde{A} \cap \widetilde{B}) = \widetilde{A}. \end{gathered} \tag{41}$$
9. The law of involution for the dual element:
$$(\widetilde{A}^D)^D = \widetilde{A}. \tag{42}$$
10. The law of Involution for the fuzzy complement:
$$\neg(\neg \widetilde{A}) = \widetilde{A}. \tag{43}$$
11. Identity:
$$\begin{gathered} \widetilde{A} \cup \emptyset = \widetilde{A} \quad \widetilde{A} \cap \Omega = \widetilde{A} \\ \widetilde{A} \cup \Omega = \Omega \quad \widetilde{A} \cap \emptyset = \emptyset. \end{gathered} \tag{44}$$
12. The law of order circulation:
$$(\widetilde{A} \subseteq \widetilde{B}) \Leftrightarrow (\neg \widetilde{B} \subseteq \neg \widetilde{A}). \tag{45}$$
13. De Morgan's laws:
$$\begin{gathered} \neg(\widetilde{A} \cup \widetilde{B}) = (\neg \widetilde{A}) \cap (\neg \widetilde{B}) \\ \neg(\widetilde{A} \cap \widetilde{B}) = (\neg \widetilde{A}) \cup (\neg \widetilde{B}). \end{gathered} \tag{46}$$

14. Duality laws for union and intersection of split subsets:

$$\begin{aligned} &(\widetilde{A} \cup \widetilde{B})^D = (\widetilde{A}^D \cap \widetilde{B}^D) \cup (\widetilde{A}^C \cap \widetilde{B}^D) \cup (\widetilde{B}^C \cap \widetilde{A}^D) \\ &(\widetilde{A} \cap \widetilde{B})^D = (A \cap \widetilde{B}^D) \cup (B \cap \widetilde{A}^D) \end{aligned} \tag{47}$$

For example, let us prove the laws 13 and 14. We have:

$$\neg(\widetilde{A} \cup \widetilde{B}) \rightleftarrows 1 - I_{\widetilde{A} \cup \widetilde{B}} = 1 - (I_{\widetilde{A}} \vee I_{\widetilde{B}}) = (1 - I_{\widetilde{A}}) \wedge (1 - I_{\widetilde{B}}) = I_{\neg \widetilde{A}} \wedge I_{\neg \widetilde{B}} \rightleftarrows (\neg \widetilde{A}) \cap (\neg \widetilde{B})$$
$$\neg(\widetilde{A} \cap \widetilde{B}) \rightleftarrows 1 - I_{\widetilde{A} \cap \widetilde{B}} = 1 - (I_{\widetilde{A}} \wedge I_{\widetilde{B}}) = (1 - I_{\widetilde{A}}) \vee (1 - I_{\widetilde{B}}) = I_{\neg \widetilde{A}} \vee I_{\neg \widetilde{B}} \rightleftarrows (\neg \widetilde{A}) \cup (\neg \widetilde{B}).$$

To prove the first law (47), let us proceed as follows. On the one hand,

$$\neg(I_{\widetilde{A}} \vee I_{\widetilde{B}}) = (I_A \vee I_B)^C \vee (I_{\widetilde{A}} \vee I_{\widetilde{B}})^D = (I_{\widetilde{A}} \vee I_{\widetilde{B}})^D \vee (I_{A^C} \wedge I_{B^C}).$$

Since $(\sup(I_{\widetilde{A}} \vee I_{\widetilde{B}})^D) \cap) \sup(I_{A^C} \wedge I_{B^C})) = \emptyset (\sup(I_{\widetilde{A}} \vee I_{\widetilde{B}})^D) \cap) \sup(I_{A^C} \wedge I_{B^C})) = \emptyset$, then

$$\neg(I_{\widetilde{A}} \vee I_{\widetilde{B}}) = (I_{\widetilde{A}} \vee I_{\widetilde{B}})^D + (I_{A^C} \wedge I_{B^C}). \tag{48}$$

On the other hand, according to (46)

$$\begin{aligned} \neg(I_{\widetilde{A}} \vee I_{\widetilde{B}}) &= I_{\neg \widetilde{A}} \wedge I_{\neg \widetilde{B}} = (I_{A^C} \vee I_{\widetilde{A}^D}) \wedge (I_{B^C} \vee I_{\widetilde{B}^D}) \\ &= (I_{A^C} \wedge I_{B^C}) \vee (I_{A^C} \wedge I_{\widetilde{B}^D}) \vee (I_{\widetilde{A}^D} \wedge I_{B^C}) \vee (I_{\widetilde{A}^D} \wedge I_{\widetilde{B}^D}). \end{aligned}$$

Similarly, to the previous case

$$(\sup(I_{A^C} \wedge I_{B^C})) \cap (\sup(I_{A^C} \wedge I_{\widetilde{B}^D}) \vee (I_{\widetilde{A}^D} \wedge I_{B^C}) \vee (I_{\widetilde{A}^D} \wedge I_{\widetilde{B}^D})) = \emptyset.$$

Therefore,

$$\neg(I_{\widetilde{A}} \vee I_{\widetilde{B}}) = (I_{A^C} \wedge I_{B^C}) + \left[(I_{A^C} \wedge I_{\widetilde{B}^D}) \vee (I_{\widetilde{A}^D} \wedge I_{B^C}) \vee (I_{\widetilde{A}^D} \wedge I_{\widetilde{B}^D})\right]. \tag{49}$$

Comparing (48) and (49), we obtain the required proof.

Let us now prove the second law (47). We have

$$\begin{aligned} (\neg(I_{\widetilde{A}} \wedge I_{\widetilde{B}})) \wedge (I_A \wedge I_B) &= (I_{\widetilde{\neg A}} \vee I_{\widetilde{\neg B}}) \wedge (I_A \wedge I_B) \\ &= ((I_{A^C} \vee I_{\widetilde{A}^D}) \vee (I_{\widetilde{B}^D} \vee I_{B^C})) \wedge (I_A \wedge I_B) \\ &= (I_{A^C} \wedge (I_A \wedge I_B)) \vee (I_{\widetilde{A}^D} \wedge (I_A \wedge I_B)) \\ &\quad \vee (I_{B^C} \wedge (I_A \wedge I_B)) \vee (I_{\widetilde{B}^D} \wedge (I_A \wedge I_B)) \\ &= (I_A \wedge I_{\widetilde{B}^D}) \vee (I_{\widetilde{A}^D} \wedge I_B). \end{aligned}$$

Consider some examples of splitting sets.

Example 1 *Let Ω be a universal set, $A, B \subseteq \Omega$.*

We have the equality $A \backslash B = A \cap B^C$. If we split subsets A and B, then the splitting of this equality, according to (5) *and* (10)*, will be as follows:*

$$\widetilde{A \backslash B} = \widetilde{A \cap B^C} = \widetilde{A} \cap \widetilde{B^C} = \widetilde{A} \cap (\emptyset : \widetilde{B}). \tag{50}$$

Note that the splitting of a difference does not depend on the splitting of a "subtrahend". In addition, $\widetilde{A \backslash B}$ is not in any relation to $\widetilde{A} \backslash \widetilde{B}$, because the last expression is not defined by us at all.

Example 2 *Splitting of a symmetric difference.*

We have:

$$\begin{aligned} A\Delta B &= (A\backslash B)\cup(B\backslash A),\\ (\widetilde{A\Delta B}) &= (\widetilde{A\backslash B})\cup(\widetilde{B\backslash A})\\ &= (\widetilde{A}\cap B^C)\cup(\widetilde{B}\cap A^C). \end{aligned} \tag{51}$$

On the other hand,

$$A\Delta B = (A\cup B)\backslash(A\cap B)$$

Therefore, for the split symmetric difference, we have such a formula

$$(\widetilde{A\Delta B}) = (\widetilde{A\cup B})\cap(A\cap B)^C = (\widetilde{A}\cup\widetilde{B})\cap(A^C\cup B^C) \tag{52}$$

According to (51) *and* (52) *we have*

$$(\widetilde{A}\cap B^C)\cup(\widetilde{B}\cap A^C) = (\widetilde{A}\cup\widetilde{B})\cap(A^C\cup B^C). \tag{53}$$

In fact, according to (40) *we have*

$$\begin{aligned} (\widetilde{A}\cup\widetilde{B})\cap(A^C\cup B^C) &= (\widetilde{A}\cap(A^C\cup B^C))\cup(\widetilde{B}\cap(A^C\cup B^C))\\ &= ((\widetilde{A}\cap A^C)\cup(\widetilde{A}\cap B^C))\cup((\widetilde{B}\cap A^C)\cup(\widetilde{B}\cap B^C))\\ &= (\widetilde{A}\cap B^C)\cup(\widetilde{B}\cap A^C). \end{aligned}$$

(51) *and* (52) *can be rewritten as*

$$\begin{aligned} (\widetilde{A\Delta B}) &= (\widetilde{A}\cap B^C)\cup(\widetilde{B}\cap A^C)\\ &= (\widetilde{A}\cup\widetilde{B})\cap(A^C\cup B^C)\\ &= (\widetilde{A}\cap(\emptyset:\widetilde{B}))\cup(\widetilde{B}\cap(\emptyset:\widetilde{A}))\\ &= (\widetilde{A}\cup\widetilde{B})\cap((\emptyset:\widetilde{A})\cup(\emptyset:\widetilde{B})). \end{aligned} \tag{54}$$

Example 3 *Given is a universal set* Ω *and the sequence of nested subsets* $\Omega\supseteq\Lambda_1\supseteq\Lambda_2\supseteq\ldots$. *The following equality is known*

$$(\Lambda_1\backslash\Lambda_2)\cup(\Lambda_2\backslash\Lambda_3)\cup\ldots = \Lambda_1\backslash\bigcap_{j=1}^{\infty}\Lambda_j. \tag{55}$$

Splitting $\Lambda_1,\Lambda_2,\ldots$ *leads, according to the previous example, to equality*

$$\left(\widetilde{\Lambda_1}\cap\Lambda_2^C\right)\cup\left(\widetilde{\Lambda_2}\cap\Lambda_3^C\right)\cup\ldots = \widetilde{\Lambda_1}\cap\left(\bigcap_{j=1}^{\infty}\Lambda_j\right)^C = \widetilde{\Lambda_1}\cap\left(\bigcup_{j=1}^{\infty}\Lambda_j^C\right) = \widetilde{\Lambda_1}\cap\left(\bigcup_{j=}^{\infty}(\emptyset:\widetilde{\Lambda_j})\right). \tag{56}$$

Provided: $\forall I_{\widetilde{\Lambda_j}}$ *is a restriction of* $I_{\widetilde{\Lambda_1}}$ *on corresponding* Λ_j, *i.e.,* $\widetilde{\Lambda_j}=\widetilde{\Lambda_1}\cap\Lambda_j$. *This must be required because equality between fuzzy subsets implies equality of membership functions of the right and left sides of equality.*

Example 4 *Given is a universal set* Ω *and sequence of its subsets* $A_1 \subseteq A_2 \subseteq \ldots \subseteq \Omega$. *As we know,*

$$\bigcup_{j=1}^{\infty} \Lambda_j = \Lambda_1 \cup (\Lambda_2 \backslash \Lambda_1) \cup (\Lambda_3 \backslash \Lambda_2) \cup \ldots \tag{57}$$

Splitting $\Lambda_1, \Lambda_2, \ldots$ *leads to the formula:*

$$\begin{aligned} \widetilde{\bigcup_{j=1}^{\infty} \Lambda_j} = \bigcup_{j=1}^{\infty} \widetilde{\Lambda}_j &= \widetilde{\Lambda_1} \cup (\widetilde{\Lambda_2} \cap \Lambda_1^C) \cup (\widetilde{\Lambda_3} \cap \Lambda_2^C) \cup \ldots \\ &= \widetilde{\Lambda_1} \cup (\widetilde{\Lambda_2} \cap (\emptyset : \widetilde{\Lambda_1})) \cup (\widetilde{\Lambda_3} \cap (\emptyset : \widetilde{\Lambda_2})) \cup \ldots \end{aligned} \tag{58}$$

provided $I_{\widetilde{\Lambda_j}} \geq I_{\widetilde{\Lambda_{j+1}}} \wedge I_{\Lambda_j}$ *or* $\widetilde{\Lambda_j} \supseteq \widetilde{\Lambda_{j+1}} \cap \Lambda_j$.

5. Some formulas related to relative pseudo-complementarity

Definition 4 *The set of all possible split sets of degree* Ω, $\mathrm{P}(\Omega)$ *is called the generalized degree* Ω *and denoted by* $\widetilde{\mathrm{P}}(\Omega)$. *The natural relation of order in the* $\widetilde{\mathrm{P}}(\Omega)$ *we define as follows:*

$$(\widetilde{A} \leq \widetilde{B}) \Leftrightarrow (\widetilde{A} \subseteq \widetilde{B}). \tag{59}$$

Ordered set $[\widetilde{\mathrm{P}}(\Omega);\, \cap,\, \cup]$ *is obviously a lattice and, by virtue of isomorphism*

$$[\widetilde{\mathrm{P}}(\Omega);\, \cap,\, \cup] \rightleftarrows [\widetilde{I};\, \wedge, \vee]$$

takes place.

Theorem 5 *Generalized degree of universal set* Ω, $\widetilde{\mathrm{P}}(\Omega)$, *is the Brewer lattice.*

See the proof in Appendix A.

Further, on the basis of (43), (46) and (47) we can write

$$(\neg\widetilde{A})^D = \widetilde{A}. \tag{60}$$

Proof.

$$\begin{aligned} (\neg\widetilde{A})^D &= (A^C \cup \widetilde{A}^D)^D \\ &= ((A^C)^C \cap (\widetilde{A}^D)^D) \cup ((A^C)^D \cap (\widetilde{A}^D)) \cup (A^C \cap (A^C)^D) \\ &= (A \cap \widetilde{A}) \cup (\emptyset \cap \widetilde{A}) \cup (A^C \cap \emptyset) \\ &= A \cap \widetilde{A}) = \widetilde{A}. \end{aligned}$$

Similarly

$$\neg(\widetilde{A}^D) = \neg(A \cap (\neg\widetilde{A})) = (\neg A) \cup (\neg(\neg\widetilde{A})) = A^C \cup \widetilde{A}. \tag{61}$$

Therefore

$$(\widetilde{A} : \widetilde{A}^D) = \begin{cases} \neg(\widetilde{A}^D), & if\ \widetilde{A}^D \nsubseteq \widetilde{A}; \\ \Omega & if\ \widetilde{A}^D \subseteq \widetilde{A}. \end{cases} \tag{62}$$

Similarly

$$(\widetilde{A}^D : \widetilde{A}) = \begin{cases} \neg\widetilde{A}, & if\ \widetilde{A} \not\subseteq \widetilde{A}^D; \\ \Omega & if\ \widetilde{A} \subseteq \widetilde{A}^D. \end{cases} \tag{63}$$

Obviously,

$$(\widetilde{A}^D : \neg\widetilde{A}) = (\neg\widetilde{A} \oplus (\neg\widetilde{A})^D)^C \vee \widetilde{A}^D = \Omega^C \cup \widetilde{A}^D = \widetilde{A}^D. \tag{64}$$

$$(\neg\widetilde{A} : \widetilde{A}^D) = \Omega. \tag{65}$$

The following formulas can be simply proved

$$(\widetilde{A} : A) = A^C \cup \widetilde{A},\ \ (A : \widetilde{A}) = \Omega, \tag{66}$$

$$\begin{gathered} (\emptyset : \widetilde{A}) = A^C, \quad (\emptyset : \widetilde{A}^C) = A\,, \\ (\emptyset : \widetilde{A}^D) = A^C, \quad (\emptyset : \neg\widetilde{A}) = \emptyset, \end{gathered} \tag{67}$$

$$(\widetilde{A} : \emptyset) = \Omega\,,\ \ (\widetilde{A} : \emptyset) \cap (\emptyset : \widetilde{A}) = A^C, \tag{68}$$

$$(\emptyset : (\emptyset : \widetilde{A})) = A, \tag{69}$$

$$(\emptyset : \widetilde{A}) = (\emptyset : (\emptyset : \widetilde{A})) = \Omega, \tag{70}$$

$$(\emptyset : \widetilde{A}) \cap (\emptyset : (\emptyset : \widetilde{A})) = \emptyset, \tag{71}$$

$$(\widetilde{B} : \widetilde{A}) \oplus (\widetilde{B} : \widetilde{A})^D = A^C \cup B. \tag{72}$$

Let us prove the last equality. According to (27) and (47) we can write: $\forall\omega \in \Omega$

$$\begin{aligned} I_{(\widetilde{B}:\widetilde{A})^D}(\omega) &= \begin{cases} (I_{A^C} \vee I_{\widetilde{B}})^D(\omega), & I_{\widetilde{A}}(\omega) > I_{\widetilde{B}}(\omega)\,; \\ 0, & I_{\widetilde{A}}(\omega) \le I_{\widetilde{B}}(\omega)\,; \end{cases} \\ &= \begin{cases} ((I_{(A^C)^D} \wedge I_{B^D}) \vee (I_{(A^C)^C} \wedge I_{\widetilde{B}^D}) \vee (I_{B^C} \wedge I_{(A^C)^D}))(\omega), & I_{\widetilde{A}}(\omega) > I_{\widetilde{B}}(\omega); \\ 0, & I_{\widetilde{A}}(\omega) \le I_{\widetilde{B}}(\omega); \end{cases} \\ &= \begin{cases} ((I_{\emptyset} \wedge I_{\widetilde{B}^D}) \vee (I_A \wedge I_{\widetilde{B}^D}) \vee (I_{B^C} \wedge I_{\emptyset}))(\omega), & I_{\widetilde{A}}(\omega) > I_{\widetilde{B}}(\omega); \\ 0, & I_{\widetilde{A}}(\omega) \le I_{\widetilde{B}}(\omega); \end{cases} \\ &= \begin{cases} (I_A \wedge I_{\widetilde{B}^D})(\omega), & I_{\widetilde{A}}(\omega) > I_{\widetilde{B}}(\omega)\,; \\ 0, & I_{\widetilde{A}}(\omega) \le I_{\widetilde{B}}(\omega)\,; \end{cases} \end{aligned}$$

and

$$\begin{aligned} (I_{(\widetilde{B}:\widetilde{A})} + I_{(\widetilde{B}:\widetilde{A})^D})(\omega) &= \begin{cases} (I_{A^C} \vee I_{\widetilde{B}})(\omega) + (I_A \wedge I_{\widetilde{B}^D})(\omega)\,, & I_{\widetilde{A}}(\omega) > I_{\widetilde{B}}(\omega)\,; \\ 1, & I_{\widetilde{A}}(\omega) \le I_{\widetilde{B}}(\omega)\,; \end{cases} \\ &= I_{A^C}(\omega) \vee I_B(\omega) \rightleftarrows A^C \vee B. \end{aligned}$$

Similarly

$$(\widetilde{A} : \widetilde{B}) \oplus (\widetilde{A} : \widetilde{B})^D = A \cup B^C. \tag{73}$$

Based on (72) and (73), we can write:

$$\Omega = [(\widetilde{A} : \widetilde{B}) \oplus (\widetilde{A} : B)^D] \cup [(\widetilde{B} : \widetilde{A}) \oplus (\widetilde{B} : \widetilde{A})^D]\,. \tag{74}$$

Also, it is obvious that for $\forall\widetilde{A} \in \widetilde{P}(\Omega)$ we can write

$$\Omega = (\widetilde{A}^D : \widetilde{A}) \oplus \widetilde{A} = (\widetilde{A} : \widetilde{A}^D) \oplus \widetilde{A}^D. \tag{75}$$

6. Some properties of the set splitting operation

Based on (29), a more general expression $\widetilde{A} \oplus \widetilde{B}$ can be obtained, which will make sense provided

$$\widetilde{B} \subseteq \neg\widetilde{A}, \text{ or } \widetilde{A} \subseteq \neg\widetilde{B}.$$

We can get the existence conditions for an expression $\widetilde{A} \oplus \widetilde{B} \oplus \widetilde{C}$,etc. Considering that such conditions are feasible for the following expressions, it is easy to prove that

$$\begin{aligned}
&(i)\quad \widetilde{A} \oplus \widetilde{B} = \widetilde{B} \oplus \widetilde{A},\\
&(ii)\quad \widetilde{A} \oplus (\widetilde{B} \oplus \widetilde{C}) = (\widetilde{A} \oplus \widetilde{B}) \oplus \widetilde{C},\\
&(iii)\quad (\widetilde{A} \oplus \widetilde{A}^D) \cap (\widetilde{B} \oplus \widetilde{B}^D) = (\widetilde{A \cap B}) \oplus (\widetilde{A \cap B})^D\\
&\qquad = (\widetilde{A} \cap \widetilde{B}) \oplus [(A \cap \widetilde{B}^D) \cup (\widetilde{A}^D \cap B)],\\
&(iv)\quad (\widetilde{A} \oplus \widetilde{A}^D) \cup (\widetilde{B} \oplus \widetilde{B}^D) = (\widetilde{A \cup B}) \oplus (\widetilde{A \cup B})^D\\
&\qquad = (\widetilde{A} \cup \widetilde{B}) \oplus [(\widetilde{A}^D \cap \widetilde{B}^D) \cup (A^C \cap \widetilde{B}^D) \cup (\widetilde{A}^D \cap B^C)],\\
&(v)\quad \widetilde{A} \oplus (\widetilde{B} \cap \widetilde{C}) = (\widetilde{A} \oplus \widetilde{B}) \cap (\widetilde{A} \oplus \widetilde{C}),\\
&(vi)\quad \widetilde{A} \oplus (\widetilde{B} \cup \widetilde{C}) = (\widetilde{A} \oplus \widetilde{B}) \cup (\widetilde{A} \oplus \widetilde{C}).
\end{aligned}$$

Let us prove the last two formulas

$$\begin{aligned}
&(v)\quad \widetilde{A} \oplus (\widetilde{B} \cap \widetilde{C}) \rightleftarrows I_{\widetilde{A}} + (I_{\widetilde{B}} \wedge I_{\widetilde{C}}) = (I_{\widetilde{A}} + I_{\widetilde{B}}) \wedge (I_{\widetilde{A}} + I_{\widetilde{C}}) \rightleftarrows (\widetilde{A} \oplus \widetilde{B}) \cap (\widetilde{A} \oplus \widetilde{C});\\
&(vi)\quad \widetilde{A} \oplus (\widetilde{B} \cup \widetilde{C}) \rightleftarrows I_{\widetilde{A}} + (I_{\widetilde{B}} \vee I_{\widetilde{C}}) = (I_{\widetilde{A}} + I_{\widetilde{B}}) \vee (I_{\widetilde{A}} + I_{\widetilde{C}}) \rightleftarrows (\widetilde{A} \oplus \widetilde{B}) \cup (\widetilde{A} \oplus \widetilde{C}).
\end{aligned}$$

Note that operations $\cup$ and $\cap$ are nondistributiveregarding $\oplus$, i.e.

$$\begin{aligned}
\widetilde{A} \cup (\widetilde{B} \oplus \widetilde{C}) \neq (\widetilde{A} \cup \widetilde{B}) \oplus (\widetilde{A} \cup \widetilde{C});\\
\widetilde{A} \cap (\widetilde{B} \oplus \widetilde{C}) \neq (\widetilde{A} \cap \widetilde{B}) \oplus (\widetilde{A} \cap \widetilde{C}).
\end{aligned}$$

Definition 5 *A fuzzy point is an object defined by the splitting of an element of a universal set Ω (an ordinary point ω):*

$$\omega_0 = \widetilde{\omega_0} \oplus \widetilde{\omega_0}^D. \tag{76}$$

I.e.

$$\widetilde{\omega_0} \rightleftarrows I_{\{\widetilde{\omega_0}\}}(\omega) = \begin{cases} a, & \omega = \omega_0 \\ 0, & \omega \neq \omega_0 \end{cases};\ \ \omega_0 \in \Omega\,,\ \ a \in [0,1]. \tag{77}$$

Theorem 6 *Let Ω be a universal set, $\omega_0 \in \Omega$, and $\widetilde{\omega_0}$ be corresponding split point, then the splitting of the universal set determined by the splitting of the point ω_0, will be a relative pseudo-completion ω_0 of $\widetilde{\omega_0}^D$ in $\widetilde{\omega_0}$*

$$\widetilde{\Omega} = \widetilde{\omega_0} : \widetilde{\omega_0}^D. \tag{78}$$

Proof The proof is elementary seen from the following transformations

$$I_{\widetilde{\Omega}} = I_{\{\widetilde{\omega_0}\}} \vee I_{\{\omega_0\}^C} = I_{\neg\{\widetilde{\omega_0}\}^D} \overrightarrow{\leftarrow} \neg\widetilde{\omega_0}^D = \widetilde{\omega_0} : \widetilde{\omega_0}^D.$$

■

Remark 2 *According to* (60) *we have:*

$$\widetilde{\Omega}^D = (\neg\widetilde{\omega_0}^D)^D = \widetilde{\omega_0}^D, \;\; (\widetilde{\omega_0} : \widetilde{\omega_0}^D) \oplus \widetilde{\omega_0}^D = \Omega. \tag{79}$$

7. ON SOME PROPERTIES OF THE MAIN IDEALS OF THE LATTICE

Consider the lattice $\widetilde{I}$. As we can see, the pseudo-completion of an arbitrary element $\widetilde{A}$ of this lattice is an element of the Boolean sublattice I in $\widetilde{I}$, designated by A^C. The set of all $\widetilde{A}$ with pseudo-completion A^C, obviously, form an ideal that we denote by $\ddot{J}_A$. For the set $A = (\emptyset : \widetilde{A^C})$ the corresponding ideal will be $\ddot{J}_{A^C}$. Denote by $\ddot{J}$ the set of all such ideals and let us order this set like this:

$$A \subseteq B \Leftrightarrow \ddot{J}_A \leq \ddot{J}_B. \tag{80}$$

Let us introduce lattice operations

$$\ddot{J}_A \wedge \ddot{J}_B = \ddot{J}_{A\cap B}, \tag{81}$$

$$\ddot{J}_A \vee \ddot{J}_B = \ddot{J}_{A\cup B}. \tag{82}$$

The set $\ddot{J}$ ordered by the above-mentioned way with operations (81) and (82) is a lattice. Obviously, there is valid the following.

Theorem 7 *The lattice $\ddot{J}$ is isomorphic to Boolean lattice I.*

Proof It is a correspondence like this $\ddot{J}_A \rightleftarrows A^C$. Therefore, due to (27) and (28) there are valid relationships

$$\ddot{J}_A \leq \ddot{J}_B \rightleftarrows A^C \geq B^C, \tag{83}$$

$$\ddot{J}_{A\cup B} = \ddot{J}_A \vee \ddot{J}_B \rightleftarrows A^C \cap B^C, \tag{84}$$

$$\ddot{J}_{A\cap B} = \ddot{J}_A \wedge \ddot{J}_B \rightleftarrows A^C \cup B^C. \tag{85}$$

$\ddot{J}_A$ is a main ideal in $\widetilde{I}$: $\widetilde{I}(A) = \ddot{J}_A$. ■

Theorem 8 *If $\ddot{J}_A$ is a main ideal in $\widetilde{I}$, generated by the element A ,then mapping on $\widetilde{X}$: $Q(\widetilde{X}) = \widetilde{X} \vee A$ is a $\vee$-endomorphism with a kernel $\ddot{J}_A$ [25].*

Proof It is easy to show that

$$Q(\widetilde{X} \vee \widetilde{Y}) = (\widetilde{X} \vee A) \vee (\widetilde{Y} \vee A) = Q(\widetilde{X}) \vee Q(\widetilde{Y})$$

Further, if $\widetilde{z} \in \ddot{J}_A$, $\widetilde{z} \leq A \Rightarrow \widetilde{z} \vee A = A \Rightarrow Q(\widetilde{z}) = A$. For $\forall \widetilde{X} \in \widetilde{I}$ there holds $Q(\widetilde{X}) = \widetilde{X} \vee A \geq A$, so if $\widetilde{z} \in \ddot{J}_A$, then $Q(\widetilde{z}) \leq Q(\widetilde{X})$. This means that A is the least element in ImQ. Thus $\ddot{J}_A = \ker Q$ is a kernel of $\vee$-endomorphism. ■

8. CONCLUSION

The article deals with the operation of splitting a crisp indicator in the dual fuzzy sets. The representations of the operations of union, intersection, Cartesian product and other operations on split indicators are also given. It is studied the lattice of split elements of the Boolean lattice of indicators *i*, where it is proved that the lattice of all split elements of this lattice $\tilde{i}$ is a Brewer lattice. A number of facts are given on the properties of this lattice. Splitting operation of a crisp set is defined, which is equivalent to splitting operation of its indicator. The main properties of this operation are given, with some proofs. The concept of the generalized degree of the universe is defined, which is the lattice of the elements obtained by splitting all the subsets of the universe. It is proved that this lattice represents a Brewer lattice. Some formulas for conditional pseudocompletion of the element of this lattice are considered. Some properties of the operation of splitting sets are given. The ideal of split elements of $\tilde{i}$ and their pseudocompletions is discussed. It is argued that this lattice is equivalent to a Boolean lattice *i*. A simple example of MADM is presented for illustration of the application of splitting operation.Future studies, aimed at multi-criteria decision-making problems, will use the results presented in this article. It is also planned to generalize the splitting operation from the fuzzy sets of Zadeh for the dual q-rung orthopair fuzzy sets.

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Appendix A.

Proof of the Theorem 2.

Proof A Brewer lattice is a lattice *l* in which, for any given elements a and b a set of all $x \in l$ such that $a \wedge x \leq b$, has the largest element $(b : a)$, called a relative pseudo-complement of a in b (in the

Brewer lattice with 0 the element $(0 : a)$ is called a pseudo-complement for a and denoted by a^*) [2].

Now, let $I_{\widetilde{A}}$, $I_{\widetilde{B}} \in \widetilde{I}$. We have to show that the set

$$\left\{I_{\widetilde{X}}\right\} = \left\{I_{\widetilde{X}} : I_{\widetilde{A}} \wedge I_{\widetilde{X}} \leq I_{\widetilde{B}},\, I_{\widetilde{X}} \in \widetilde{I}\right\} \tag{A-1}$$

Has the largest element $(I_{\widetilde{B}} : I_{\widetilde{A}})$. On the basis of lattice "inequality" in (A-1) for $\forall\omega \in \Omega$ the following inequalities are valid:

$$I_{\widetilde{A}}(\omega) \wedge I_{\widetilde{X}}(\omega) \leq I_{\widetilde{B}}(\omega)\,.$$

Let us prove that, $\forall\omega \in \Omega$

$$(I_{\widetilde{B}} : I_{\widetilde{A}})(\omega) = \begin{cases} 1, & I_{\widetilde{A}}(\omega) \leq I_{\widetilde{B}}(\omega) \\ I_{A^C}(\omega) \vee I_{\widetilde{B}}(\omega), & I_{\widetilde{A}}(\omega) > I_{\widetilde{B}}(\omega) \end{cases} \tag{A-2}$$

First, let us show that $(I_{\widetilde{B}} : I_{\widetilde{A}})(\omega) \in \left\{I_{\widetilde{X}}\right\}$. In fact, $\forall\omega \in \Omega$

$$I_{\widetilde{A}}(\omega) \wedge \begin{cases} 1, & I_{\widetilde{A}}(\omega) \leq I_{\widetilde{B}}(\omega) \\ I_{A^C}(\omega) \vee I_{\widetilde{B}}(\omega), & I_{\widetilde{A}}(\omega) > I_{\widetilde{B}}(\omega) \end{cases} = \begin{cases} I_{\widetilde{A}}(\omega), & I_{\widetilde{A}}(\omega) \leq I_{\widetilde{B}}(\omega) \\ I_{A^C}(\omega) \wedge I_{\widetilde{B}}(\omega), & I_{\widetilde{A}}(\omega) > I_{\widetilde{B}}(\omega). \end{cases}$$

To show that (A-2) is the largest element of the subset $\left\{I_{\widetilde{X}}\right\}$, note that if $I_{\widetilde{y}} \in \left\{I_{\widetilde{X}}\right\}$, then $I_{\widetilde{y}} \vee I_{A^C} \in \left\{I_{\widetilde{X}}\right\}$, and any element of $\left\{I_{\widetilde{X}}\right\}$ can be taken as $I_{\widetilde{y}}$.

Consider the intersection (A-2) with $(I_{\widetilde{y}} \vee I_{A^C})$ at each point of Ω. We have $\forall\omega \in \Omega$

$$(I_{\widetilde{y}}(\omega) \vee I_{A^C}(\omega)) = \begin{cases} 1, & I_{\widetilde{A}}(\omega) \leq I_{\widetilde{B}}(\omega) \\ I_{A^C}(\omega) \vee I_{\widetilde{B}}(\omega), & I_{\widetilde{A}}(\omega) > I_{\widetilde{B}}(\omega) \end{cases}.$$

Clear that when $I_{\widetilde{A}}(\omega) \leq I_{\widetilde{B}}(\omega)$, then

$$(I_{\widetilde{y}} \vee I_{A^C})(\omega) \leq (I_{\widetilde{B}} : I_{\widetilde{A}})(\omega).$$

In the opposite case, if we consider elemental inequality $(I_{\widetilde{y}}(\omega) \vee I_{A^C}(\omega)) \wedge I_{\widetilde{B}}(\omega) \leq I_{\widetilde{B}}(\omega)$, we will have:

$$\begin{aligned} &(I_{\widetilde{y}}(\omega) \vee I_{A^C}(\omega)) \wedge (I_{A^C}(\omega) \vee I_{\widetilde{B}}(\omega)) \\ &\quad = ((I_{\widetilde{y}}(\omega) \vee I_{A^C}(\omega)) \wedge I_{A^C}(\omega)) \vee ((I_{\widetilde{y}}(\omega) \vee I_{A^C}(\omega)) \wedge I_{\widetilde{B}}(\omega)) \\ &\quad = I_{A^C}(\omega) \vee ((I_{\widetilde{y}}(\omega) \vee I_{A^C}(\omega)) \wedge I_{\widetilde{B}}(\omega)) \\ &\quad \leq I_{A^C}(\omega) \vee I_{\widetilde{B}}(\omega). \end{aligned}$$

Finally, $\forall\omega \in \Omega$ and $\forall I_{\widetilde{X}} \in \left\{I_{\widetilde{X}}\right\}$

$$I_{\widetilde{X}}(\omega) \leq (I_{\widetilde{B}} : I_{\widetilde{A}})(\omega) = \begin{cases} 1, & I_{\widetilde{A}}(\omega) \leq I_{\widetilde{B}}(\omega); \\ I_{A^C}(\omega) \vee I_{\widetilde{B}}(\omega), & I_{\widetilde{A}}(\omega) > I_{\widetilde{B}}(\omega). \end{cases}$$

Theorem is proved. ■

Proof of the Theorem 3.

Proof

$(i)\ \ I_{\widetilde{A}} \le I_{\widetilde{B}} \Leftrightarrow (I_{\widetilde{A}}(\omega) \le I_{\widetilde{B}}(\omega)\,,\ \ \forall\omega\in\Omega) \Leftrightarrow\Leftrightarrow (I_{B^C}(\omega) \le I_{A^C}(\omega)) \Leftrightarrow (I_\emptyset : I_{\widetilde{B}}) \le (I_\emptyset : I_{\widetilde{A}}).$

$(ii)\ \ (I_\emptyset : (I_\emptyset : I_{\widetilde{A}})) = (I_\emptyset : I_{A^C}) = I_A.$

From inequality $I_{\widetilde{A}} \le I_A$ is elementarily provable.

$$(iii)\ \ (I_\emptyset : (I_\emptyset : (I_\emptyset : I_{\widetilde{A}}))) = I_{A^C} = (I_\emptyset : I_{\widetilde{A}})\,.$$

$$\begin{aligned}(iv)\ \ (I_\emptyset : (I_{\widetilde{A}} \vee I_{\widetilde{B}})) &= ((I_{\widetilde{A}} \vee I_{\widetilde{B}}) + (I_{\widetilde{A}} \vee I_{\widetilde{B}})^D)^C \\ &= (I_{\widetilde{A\cup B}} + I_{\widetilde{A\cup B}^D})^C = I_{(A\cup B)^C} = I_{A^C\cap B^C} = I_{A^C} \wedge I_{B^C} = (I_\emptyset : I_{\widetilde{A}}) \wedge (I_\emptyset : I_{\widetilde{B}})\,.\end{aligned}$$

$$\begin{aligned}(v)\ \ (I_\emptyset : (I_{\widetilde{A}} \wedge I_{\widetilde{B}})) &= ((I_{\widetilde{A}} \wedge I_{\widetilde{B}}) + (I_{\widetilde{A}} \wedge I_{\widetilde{B}})^D)^C \\ &= (I_{\widetilde{A\cap B}} + I_{\widetilde{A\cap B}^D})^C = I_{(A\cap B)^C} = I_{A^C\cup B^C} = I_{A^C} \vee I_{B^C} = (I_\emptyset : I_{\widetilde{A}}) \vee (I_\emptyset : I_{\widetilde{B}})\,.\end{aligned}$$

■

Proof of the Theorem 4.

Proof For $\forall\omega\in\Omega$ we have

$$\begin{aligned}(i)\ \ (I_{\widetilde{A}} : I_{\widetilde{B}})(\omega) \wedge I_{\widetilde{A}}(\omega) &= \left(\begin{cases} I_{B^C}(\omega) \vee I_{\widetilde{A}}(\omega), & I_{\widetilde{B}}(\omega) > I_{\widetilde{A}}(\omega) \\ 1, & I_{\widetilde{B}}(\omega) \le I_{\widetilde{A}}(\omega)\end{cases}\right) \wedge I_{\widetilde{A}}(\omega) \\ &= \begin{cases} (I_{B^C}(\omega) \vee I_{\widetilde{A}}(\omega)) \wedge I_{\widetilde{A}}(\omega), & I_{\widetilde{B}}(\omega) > I_{\widetilde{A}}(\omega) \\ 1 \wedge I_{\widetilde{A}}(\omega), & I_{\widetilde{B}}(\omega) \le I_{\widetilde{A}}(\omega)\end{cases} \\ &= I_{\widetilde{A}}(\omega), \quad \forall\omega\in\Omega.\end{aligned}$$

$$\begin{aligned}(ii)\ \ ((I_{\widetilde{A}} : I_{\widetilde{B}})(\omega) \wedge I_{\widetilde{B}}(\omega)) &= \left(\begin{cases} I_{B^C}(\omega) \vee I_{\widetilde{A}}(\omega), & I_{\widetilde{B}}(\omega) > I_{\widetilde{A}}(\omega) \\ 1, & I_{\widetilde{B}}(\omega) \le I_{\widetilde{A}}(\omega)\end{cases}\right) \wedge I_{\widetilde{B}}(\omega) \\ &= \begin{cases} (I_{B^C}(\omega) \vee I_{\widetilde{A}}(\omega) \wedge I_{\widetilde{B}}(\omega), & I_{\widetilde{B}}(\omega) > I_{\widetilde{A}}(\omega) \\ 1 \wedge I_{\widetilde{B}}(\omega), & I_{\widetilde{B}}(\omega) \le I_{\widetilde{A}}(\omega)\end{cases} \\ &= I_{\widetilde{A}}(\omega) \wedge I_{\widetilde{B}}(\omega)\,, \quad \forall\omega\in\Omega.\end{aligned}$$

$$\begin{aligned}(iii)\ \ ((I_{\widetilde{A}} \wedge I_{\widetilde{B}})(\omega) : I_{\widetilde{C}}(\omega)) &= \begin{cases} I_{\widetilde{C}}(\omega) \vee (I_{A^C}(\omega) \wedge I_{\widetilde{B}}(\omega),)\,, & I_{\widetilde{C}}(\omega) > (I_{\widetilde{A}}(\omega) \wedge I_{\widetilde{B}}(\omega)); \\ 1\,, & I_{\widetilde{C}}(\omega) \le (I_{\widetilde{A}}(\omega) \wedge I_{\widetilde{B}}(\omega));\end{cases} \\ &= \begin{cases} (I_{\widetilde{C}}(\omega) \vee I_{A^C}(\omega)) \\ \quad \wedge (I_{\widetilde{C}}(\omega) \vee I_{\widetilde{B}}(\omega))\,, & I_{\widetilde{C}}(\omega) > (I_{\widetilde{A}}(\omega) \wedge I_{\widetilde{B}}(\omega)); \\ 1\,, & I_{\widetilde{C}}(\omega) \le (I_{\widetilde{A}}(\omega) \wedge I_{\widetilde{B}}(\omega)).\end{cases}\end{aligned}$$

Because $I_{\widetilde{C}}(\omega) \le (I_{\widetilde{A}}(\omega) \wedge I_{\widetilde{B}}(\omega))$, then $I_{\widetilde{C}}(\omega) \le I_{\widetilde{A}}(\omega)\,, I_{\widetilde{B}}(\omega)$. Therefore, for $\forall\omega\in\Omega$ we have

$$((I_{\widetilde{A}} \wedge I_{\widetilde{B}}) : I_{\widetilde{C}})(\omega) = (I_{\widetilde{A}} : I_{\widetilde{C}})(\omega) \wedge (I_{\widetilde{B}} : I_{\widetilde{C}})(\omega).$$

$$(iv)\ \ (I_{\widetilde{A}}:(I_{\widetilde{B}}\vee I_{\widetilde{C}}))(\omega)=\begin{cases}((I_{\widetilde{B}}(\omega)\vee I_{\widetilde{C}}(\omega)+(I_{\widetilde{B}}(\omega)\\ \quad\vee I_{\widetilde{C}}(\omega)^{D})^{C}\vee I_{\widetilde{A}}(\omega), & I_{\widetilde{B}}(\omega)\vee I_{\widetilde{C}}(\omega)>I_{\widetilde{A}}(\omega);\\ 1, & I_{\widetilde{B}}(\omega)\vee I_{\widetilde{C}}(\omega)\le I_{\widetilde{A}}(\omega);\end{cases}$$
$$=\begin{cases}(I_{\widetilde{B\cup C}}(\omega)+I_{\widetilde{B\cup C}^{D}}(\omega))^{C}\vee I_{\widetilde{A}}(\omega), & I_{\widetilde{B}}(\omega)\vee I_{\widetilde{C}}(\omega)>I_{\widetilde{A}}(\omega)\,;\\ 1, & I_{\widetilde{B}}(\omega)\vee I_{\widetilde{C}}(\omega)\le I_{\widetilde{A}}(\omega).\end{cases}$$

Since $I_{\widetilde{B}}(\omega)\vee I_{\widetilde{C}}(\omega)\le I_{\widetilde{A}}(\omega)$, then $I_{\widetilde{B}}(\omega),\ I_{\widetilde{C}}(\omega)\le I_{\widetilde{A}}(\omega)$ and for $\forall\omega\in\Omega$

$$(I_{\widetilde{A}}:(I_{\widetilde{B}}\vee I_{\widetilde{C}}))(\omega)=\begin{cases}(I_{B^{C}}(\omega)\wedge I_{C^{C}}(\omega))\vee I_{\widetilde{A}}(\omega), & I_{\widetilde{B}}(\omega)\vee I_{\widetilde{C}}(\omega)>I_{\widetilde{A}}(\omega);\\ 1, & I_{\widetilde{B}}(\omega)\vee I_{\widetilde{C}}(\omega)\le I_{\widetilde{A}}(\omega);\end{cases}$$
$$=\begin{cases}(I_{B^{C}}(\omega)\vee I_{\widetilde{A}}(\omega))\wedge I_{C^{C}}(\omega)\vee I_{\widetilde{A}}(\omega), & I_{\widetilde{B}}(\omega)\vee I_{\widetilde{C}}(\omega)>I_{\widetilde{A}}(\omega);\\ 1, & I_{\widetilde{B}}(\omega)\vee I_{\widetilde{C}}(\omega)\le I_{\widetilde{A}}(\omega);\end{cases}$$
$$=(I_{\widetilde{A}}:I_{\widetilde{B}})(\omega)\wedge(I_{\widetilde{A}}:I_{\widetilde{C}})(\omega)\,.$$

Theorem is proved. ■

Proof of the Theorem 5.

Proof Let for $\forall\,\widetilde{A}\in\widetilde{P}(\Omega)$ a fuzzy subset $\widetilde{A_{>\frac{1}{2}}}\subseteq\widetilde{A}$ such that, its membership function is $>\frac{1}{2}$ and for $\widetilde{A_{\le\frac{1}{2}}}\subseteq\widetilde{A}$ is less or equal to $\frac{1}{2}$ $\left(\widetilde{A_{>\frac{1}{2}}}\cup\widetilde{A_{\le\frac{1}{2}}}=\widetilde{A}\right)$, then

$$(\widetilde{A}^{D}:\widetilde{A})=A^{C}_{>\frac{1}{2}}\cup\widetilde{A}^{D}. \tag{A-3}$$

In fact, according to (27)

$$\begin{aligned}(\widetilde{A}^{D}:\widetilde{A})\rightleftarrows(I_{\widetilde{A}^{D}}:I_{\widetilde{A}})&=(I_{\widetilde{A}^{D}}:(I_{\widetilde{A_{>\frac{1}{2}}}}\cup I_{\widetilde{A_{\le\frac{1}{2}}}}))\\&=(I_{\widetilde{A}^{D}}:I_{\widetilde{A_{>\frac{1}{2}}}})\wedge(I_{\widetilde{A}^{D}}:I_{\widetilde{A_{\le\frac{1}{2}}}})\\&=(I_{A^{C}_{>\frac{1}{2}}}\vee I_{\widetilde{A}^{D}})\wedge I_{\Omega}=I_{A^{C}_{>\frac{1}{2}}}\vee I_{A^{D}}.\end{aligned}$$

Reasoning similarly, we get

$$(\widetilde{A}:\widetilde{A}^{D})=A^{C}_{>\frac{1}{2}}\cup\widetilde{A}. \tag{A-4}$$

Since

$$\neg\widetilde{A}=A^{C}\cup\widetilde{A}^{D} \tag{A-5}$$

then

$$\widetilde{A}^{D}=A\cap(\widetilde{\neg A}). \tag{A-6}$$

Theorem is proved. ■

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